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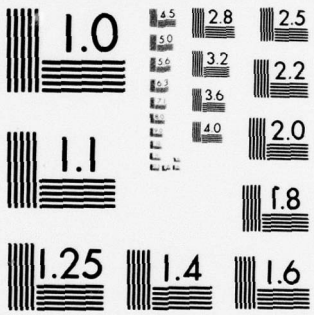


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# System Theory Aspects of Multi-Body Dynamics

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This technical report has been reviewed and is approved for publication. Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

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The spacecraft attitude dynamics and control problem is introduced as a broad class of nonlinear systems. Some of the basic properties of these systems are described from a system theory point of view. Various system theory concepts and research topics which have applicability to this class of systems are identified and briefly described. The subject of multi-body dynamics is presented in a vector space setting and is related to system theory concepts.

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## PREFACE

This report is part of a continuing effort, in the Control Analysis Department of The Aerospace Corporation, of developing techniques for analyzing and simulating the control and estimation of the dynamics and kinematics of multi-body and deformable spacecraft. The following are previous related reports:

- W. Jerkovsky, "The Structure of Multi-Body Dynamics Equations", Aerospace Corp. Rept. No. TR-0077(2901-03)-6, 1 June 1977. Presented at the AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, 13-15 June 1977. To be published in AIAA Journal of Guidance and Control, Vol. 1, No. 3, May/June 1978.
- W. Jerkovsky, "Exact Equations of Motion for a Deformable Body", Aerospace Corp. Rept. No. TR-0077(2901-03)-4, 29 March 1977.
- R.K. Williamson, "Some Aspects of the Dynamics and Stability of Spinning Flexible Spacecraft", Aerospace Corp. Rept. No. TR-0077(2901-03)-8, 12 September 1977.
- W. J. Russell, "Dynamic Analysis of the Communication Satellites of the Future", AIAA Paper No. 76-261, AIAA/CASI 6th Communication Satellite System Conference, Montreal, April 1976.

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The body of this report (excluding the Appendix) has been submitted for presentation at the IEEE Conference on Decision and Control, which is to be held on 10-12 January 1979 in San Diego, California.

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# I. INTRODUCTION

During the last two decades multi-body dynamics has been an active research subject in the field of spacecraft attitude dynamics and control, and in the field of mechanisms and manipulators. Most of this research has been oriented toward developing computationally efficient techniques for solving (or "simulating") a set of nonlinear dynamics equations. Thus, many different forms of the equations of motion have been obtained, and studies have been performed on the equivalences and relative merits of various alternative formulations [1] - [3] . All of the various formulations can be shown to be equivalent in the sense that they can be obtained from each other via nonsingular linear transformations.

The purpose of this paper is to identify some aspects of linear and nonlinear system theory which have direct application to multi-body dynamics. Classical dynamics and system theory have common roots in the transformation theory of ordinary differential equations; consequently it is not surprising that many system theory concepts have applications in classical dynamics in general, and in multi-body dynamics in particular.

This paper is aimed at system theorists interested in applications to classical dynamics, and at dynamicists interested in system theory techniques. In order to have a common background for discussion, the paper first describes the relevant aspects of a typical spacecraft attitude dynamics and control system. This is then followed by a discussion of some aspects of system theory which can meaningfully be applied to the spacecraft attitude dynamics problem. The discussion is in part informal since the intent is to demonstrate applicability of the concepts rather than to prove concrete results. The subsequent

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section is a description of multi-body dynamics from a vector space point of view. Constraints on a system are related to uncontrollability.

Numerous references to applicable system theory are provided for the interested dynamicist, and some key multi-body dynamics references are provided for the interested system theorist.

It is hoped that this paper will contribute to the cross-fertilization of the fields of system theory and spacecraft attitude dynamics. Much future work remains to be done in this area. The problems are expected to continue into the next century as larger and larger spacecraft (some several miles in size) are being considered and studied.

The motivation for undertaking the study reported herein came from the desire to find computationally efficient spacecraft attitude dynamics formulations. This required a systematic study of the structure of various alternative formulations [3]. As demonstrated in this paper, many concepts of system theory are natural tools in this study.

## II. SPACECRAFT ATTITUDE DYNAMICS AND CONTROL SYSTEMS

For the present purposes, a typical spacecraft attitude dynamics and control system can be considered to consist of three parts: dynamics, kinematics, and controller. The dynamics and kinematics together form the plant; this plant is a "given" which the control systems engineer normally does not change. The task then is to design a feedback controller such that the combined system of plant and controller has acceptable performance. A controller consists of sensors, data processors, and actuators; these components perform the functions of measurement, computation and decision, and exerting force (or torque) on the plant, respectively.

### A. Controller

Controllers are always nonlinear because real physical sensors and actuators are always nonlinear; in addition, usually it is also desireable to use nonlinear, and perhaps variable-structure, data processing. With only a small risk of oversimplifying matters, it can be said that there are as many different spacecraft controllers as there are different spacecraft, or as there are spacecraft control systems designers. However, often a controller can be modelled with a nonlinear finite-dimensional state equation, $\dot{e} = \theta(e, q, t)$, and a nonlinear output equation, $K = \psi(e, q, t)$; here $e$ is the controller state variable, $q$ is the plant output, $K$ is the controller output (force or torque), and $t$ is the time. Sometimes a portion of the controller is discrete rather than continuous. Occasionally the continuous portion of the controller is neglected and the entire controller is modelled as discrete.

## B. Dynamics

Whereas the form of the controller equations are at the control system designer's discretion, the form of the plant equations is not. However, the interesting fact is that the plant equations always have the same form (except at various discrete times the order (dimension) of these equations may change). The plant equations can be written as

$$\dot{G} = \gamma G + K \tag{2.1a}$$

$$g = \nu G \tag{2.1b}$$

where the dynamics state variable, G, is the system momentum, and the dynamics output variable, g, is the system velocity. G and g are elements of linear vector spaces, and therefore they are linearly related:

$$G = \mu g \tag{2.2}$$

where $\mu$ is the system mass (or mass/inertia). $\mu$ is always positive definite symmetric, and therefore it has a positive definite symmetric inverse, $\nu = \mu^{-1}$. Since g is related to G via a nonsingular linear transformation, it follows that g itself can be used as the dynamics state variable. The plant equations can then be written as

$$\dot{g} = -\gamma^t g + \nu K \tag{2.3a}$$

$$g = g \tag{2.3b}$$

where $\gamma^t$ is the transpose of $\gamma$. $\gamma$ is called the system Cartan operator [4]. Apparently, Equations (2.1) and (2.3) are formally

duals or adjoints of each other [5] - [6], since $\nu$ is symmetric ($\nu = \nu^t$). However, the definition of dual or adjoint equations given in linear system theory does not apply since equations (2.1) and (2.3) are not linear equations. The nonlinearities in these equations are due to the facts that $\gamma$ depends (linearly) on G or g, and both $\gamma$ and $\nu$ depend (nonlinearly) on the position $p$. But equations (2.1) and (2.3) are truly duals of each other for reasons whose roots lie in linear algebra (or tensor analysis): G and g represent the same physical quantity in different vector spaces which are the duals of each other. Let $\mathcal{V}$ be the velocity space so that $g \in \mathcal{V}$; then the dual space $\mathcal{V}^*$ is the momentum space so that $G \in \mathcal{V}^*$. The relationships between $\mathcal{V}$ and $\mathcal{V}^*$ can also be expressed as

$$\mathcal{V}^* = \mu \mathcal{V} \tag{2.4a}$$

$$\mathcal{V} = \nu \mathcal{V}^* \tag{2.4b}$$

Thus, $\mu$ maps $\mathcal{V}$ into its dual, and $\nu$ maps $\mathcal{V}^*$ into its dual. The vector spaces $\mathcal{V}$ and $\mathcal{V}^*$ are turned into inner product spaces by defining inner products as follows [7]

$$\langle g, g' \rangle_{\mathcal{V}} = g^t \mu g' \tag{2.5a}$$

$$\langle G, G' \rangle_{\mathcal{V}^*} = G^t \nu G' \tag{2.5b}$$

where $g, g' \in \mathcal{V}$ and $G, G' \in \mathcal{V}^*$. But the kinetic energy T can be expressed as [3] - [4]

$$T = \frac{1}{2} G^t g = \frac{1}{2} g^t \mu g = \frac{1}{2} G^t \nu G \tag{2.6}$$

Combining equations (2.5) and (2.6) now yields

$$2T = \langle g, g \rangle_{\mathcal{V}} = \langle G, G \rangle_{\mathcal{V}^*} \tag{2.7}$$

In the language of tensor analysis [8], the momentum G is also referred to as the covariant velocity, and the velocity g is also referred to as the contravariant momentum. $\mu$ is the covariant metric tensor, and its inverse $\nu$ is the contravariant metric tensor.

The force K can also be referred to as the covariant force; the contravariant force k is defined by

$$k = \nu K \quad \text{or} \quad K = \mu k \tag{2.8}$$

Evidently, these two equations are analogous to equations (2.1b) and (2.2). The time derivative of the kinetic energy can now be expressed as [3]

$$\dot{T} = K^t g = k^t G \tag{2.9}$$

Note that $\dot{T} = 0$, or T is constant, when $K = 0$. In the language of tensor analysis, T and $\dot{T}$ are invariants since they are equal to the scalar product of a contravariant and covariant vector.

The Cartan operator $\gamma$ not only appears in the time derivative of G and g, but it also appears in the time derivatives of $\mu$ and $\nu$ as follows:

$$\dot{\mu} = \gamma\mu + \mu\gamma^t \tag{2.10a}$$

$$\dot{\nu} = -\gamma^t\nu - \nu\gamma \tag{2.10b}$$

These equations are useful in relating equations (2.1a) and (2.3a). For example, differentiating $G = \mu g$ with respect to time yields

$\dot{G} = \dot{\mu} g + \mu \dot{g}$ ; substituting for $\dot{\mu}$ from (2.10a) and for $\dot{g}$ from (2.3a) then yields $\dot{G}$ as in (2.1a). Equations (2.10) are formally similar to the linear matrix equations which appear in linear system theory (see [5], pp. 58-62); thus, equation (2.10b) can be considered to be the adjoint equation associated with (2.10a).

All the dynamics equations given thus far have been stated without proof. These equations can be derived from Newton's law for a particle in a system of particles. However, all the dynamics equations can easily be interpreted by using the tensor analysis concept of a covariant derivative. Let $\overset{\square}{G}$ and $\overset{\square}{g}$ be defined by

$$\overset{\square}{G} = \dot{G} - \gamma G \tag{2.11a}$$

$$\overset{\square}{g} = \dot{g} + \gamma^t g \tag{2.11b}$$

and let $\overset{\square}{\mu}$ and $\overset{\square}{\nu}$ be defined by

$$\overset{\square}{\mu} = \dot{\mu} - \gamma\mu - \mu\gamma^t \tag{2.12a}$$

$$\overset{\square}{\nu} = \dot{\nu} + \gamma^t \nu + \nu\gamma \tag{2.12b}$$

A quantity with an open square dot on top will be called the covariant time derivative of that quantity. Equations (2.11) and (2.12) follow from the covariant differentiation equations of tensor analysis when it is remembered that G is covariant of degree one, g is contravariant of degree one, $\mu$ is covariant of degree two, and $\nu$ is contravariant of degree two. The elements of $\gamma$ can be expanded linearly in terms of the components of g, the coefficients in these expansions being the components of the affine connection. Using these covariant time derivatives, equations (2.1a), (2.3a), and (2.10) become simply

$$\overset{\square}{G} = K \tag{2.13a}$$

$$\overset{\square}{g} = k \tag{2.13b}$$

and

$$\overset{\square}{\mu} = 0 \tag{2.14a}$$

$$\overset{\square}{\nu} = 0 \tag{2.14b}$$

Equations (2.13) state simply that the covariant time derivative of the covariant velocity is equal to the covariant force, and the covariant time derivative of the contravariant velocity is equal to the contravariant force. Equations (2.14) state that the covariant time derivative of the metric tensor and its inverse are zero. These results are well known in tensor analysis [9] - [10]. Note that covariant differentiation is a derivation [11] in the sense that it follows the rule of Leibnitz: $G = \mu g$ implies that $\overset{\square}{G} = \overset{\square}{\mu} g + \mu \overset{\square}{g}$ ; from this and equations (2.13) and (2.14) there now follows that $K = \mu k$ , in agreement with (2.8).

It is evident from Equations (2.1) to (2.14) that the dynamics equations can be put into a very neat and simple form. It is, of course, also possible to put these equations into more complicated forms, such as introducing coordinates with respect to a time-varying reference; then additional terms arise in the expressions for the covariant derivatives and in the equations of motion, and the kinetic energy is no longer a quadratic in velocity or momentum. However, the resulting more complicated equations are not any more general, even though they may be more convenient for some particular purpose.

The dynamics equations which have been presented are not in a form which are familiar to most dynamicists. In fact, it is usually not convenient to obtain the Cartan operator $\gamma$ as a prelude

to obtaining the equations of motion. However, if $\gamma$ is found, it usually can be decomposed as $\gamma = \gamma_G + \gamma'_G$ where $\gamma'_G G = 0$ so that $\gamma G = \gamma_G G$. Similarly, $\gamma$ can be decomposed as $\gamma = \gamma_g + \gamma'_g$ where $\gamma'^t_g g = 0$ so that $\gamma^t g = \gamma^t_g g$. Thus, only $\gamma_G G$ appears in the momentum formulation equation of motion, and only $\gamma^t_g g$ appears in the velocity formulation equation; consequently, it is often not evident that the <u>same</u> Cartan operator $\gamma$ appears in $\dot{G}$ and $\dot{g}$, <u>and</u> in $\dot{\mu}$ and $\dot{\nu}$.

A method of finding $\gamma$ for any particular problem is described in Section IV. B. Briefly, the method consists of first expressing the equations of motion in terms of variables for which $\gamma$ is known, and then transforming to more convenient variables. Thus, for example, the $\gamma$ for a single particle (which obeys Newton's laws of motion) is zero, and the $\gamma$ for a system of particles is therefore also zero. By considering a rigid (or perhaps flexible) body to consist of N particles, the $\gamma$ for the rigid (or flexible) body can be found. From this $\gamma$ the $\gamma$ for any arbitrary system of bodies can be found.

## C. Kinematics

The system momentum, G, and system velocity, g, are vectors, but the system position, $p$, is a point. Thus, G and g are elements of a vector space, whereas $p$ is an element of a manifold $\mathcal{M}$ [12] - [13]. If the plant has n degrees of freedom, then the linear vector spaces $\mathcal{V}$ and $\mathcal{V}^*$ have dimension n, and so does the manifold $\mathcal{M}$. Therefore, g and G have n independent components, and the point $p$ is specified via n coordinates, $q_i$, for i = 1 to n. Let q be the n-element column matrix which has $q_i$ as the ith element, and let g now represent an n-element column matrix whose ith element is the ith component of the system velocity vector (with respect to some basis). The time derivative of q can now be linearly related to g:

$$\dot{q} = \alpha g \tag{2.15}$$

where $\alpha$ depends (nonlinearly) on the point: $\alpha = \alpha(p)$, or $\alpha = \alpha(q)$. Note that $\dot{q}$ is an element of a vector space [this vector space is the tangent space to the manifold $\mathcal{M}$ at the point $p$] even though q itself is not an element of a vector space.

Equation (2.15) is really a change of coordinates in the velocity space $\mathcal{V}$. Hence, $\alpha$ is invertible; calling the inverse $\beta$ yields

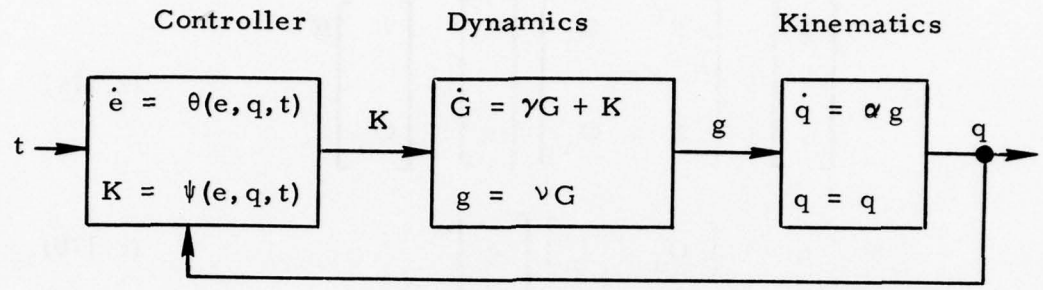
$$g = \beta \dot{q} \tag{2.16}$$

where $\beta = \alpha^{-1}$. If the components of the velocity are time derivatives of coordinates [ e.g., when $\alpha$ is the identity ], then g is called a holonomic velocity [ strictly, the components of g are holonomic velocity components ] ; holonomic components are also called integrable components. If the components of the velocity are not time derivatives of coordinates [ i.e., the components are not integrable ], then they are called nonholonomic velocity components. Interesting discussions of nonholonomic dynamics equations (also known as Boltzmann-Hamel equations and as Lagrange equations in quasi-coordinates) can be found in [14]-[15].

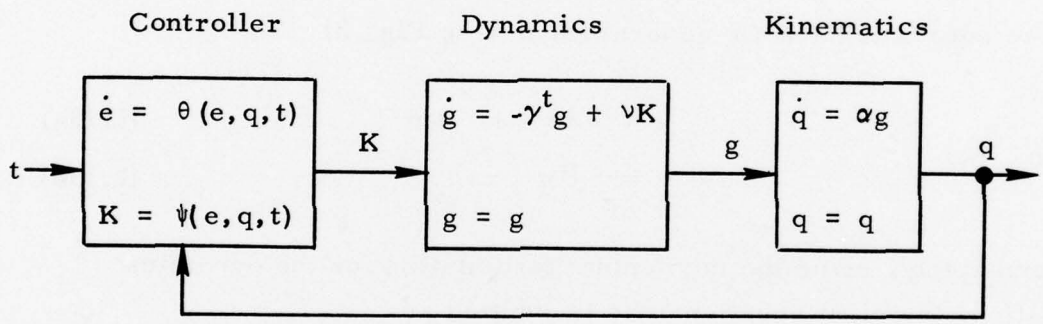
After a basis is introduced $\mu$, $\nu$, and $\gamma$ become $n \times n$ matrices. $\mu$ and $\nu$ depend nonlinearly on q: $\mu = \mu(q)$ and $\nu = \nu(q)$; $\gamma$ depends nonlinearly on q and linearly on g or G: $\gamma = \gamma(q, g)$ or $\gamma = \gamma(q, G)$.

## D. Plant

Using the velocity formulation for the dynamics equation, the plant equations can now be written in the form [see Fig. 1]




a) Momentum Formulation




b) Velocity Formulation

Figure 1. Block Diagram of Spacecraft Attitude Controller, Dynamics, and Kinematics

$$\begin{bmatrix} \dot{g} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -\gamma^t & O_n \\ \alpha & O_n \end{bmatrix} \begin{bmatrix} g \\ q \end{bmatrix} + \begin{bmatrix} \nu \\ O_n \end{bmatrix} K \tag{2.17a}$$

$$q = \begin{bmatrix} O_n & 1_n \end{bmatrix} \begin{bmatrix} g \\ q \end{bmatrix} \tag{2.17b}$$

These equations have the general form (see Fig. 2)

$$\dot{x} = \mathcal{A} x + \mathcal{B} u \tag{2.18a}$$

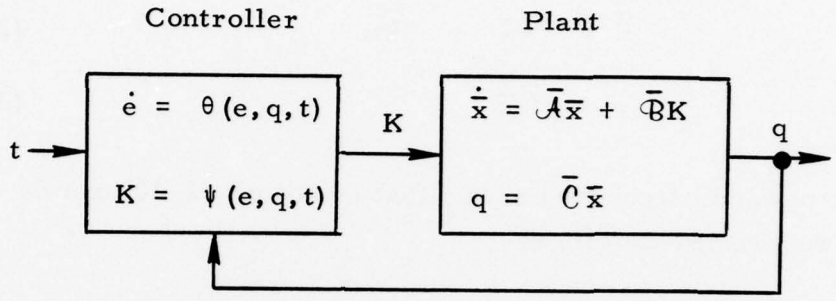
$$y = \mathcal{C} x \tag{2.18b}$$

Alternatively, using the momentum formulation for the dynamics equation, the plant equations can be written as

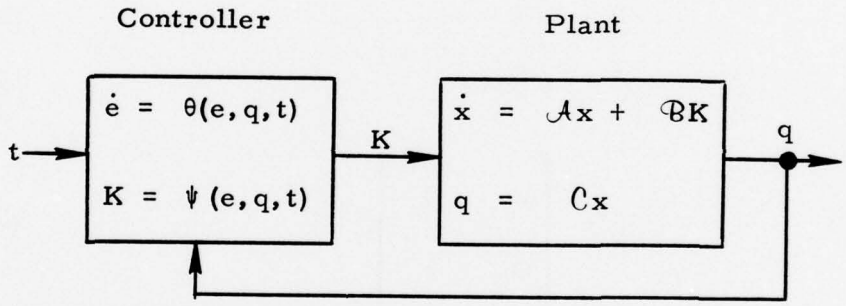
$$\begin{bmatrix} \dot{G} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \gamma & O_n \\ \bar{\alpha} & O_n \end{bmatrix} \begin{bmatrix} G \\ q \end{bmatrix} + \begin{bmatrix} 1_n \\ O_n \end{bmatrix} K \tag{2.19a}$$

$$\bar{\alpha} = \alpha \nu \tag{2.19b}$$

$$q = \begin{bmatrix} O_n & 1_n \end{bmatrix} \begin{bmatrix} G \\ q \end{bmatrix} \tag{2.19c}$$




a) Momentum Formulation




b) Velocity Formulation

Figure 2. Block Diagram of Spacecraft Attitude Controller and Plant

which have the form

$$\dot{\bar{x}} = \bar{\mathcal{A}}\bar{x} + \bar{\mathcal{B}}u \tag{2.20a}$$

$$y = \bar{\mathcal{C}}\bar{x} \tag{2.20b}$$

A straightforward calculation shows that equations (2-20) can be obtained from (2.18) as follows

$$\bar{x} = \mathcal{P}x \tag{2.21a}$$

$$\bar{\mathcal{A}} = (\mathcal{P}\mathcal{A} + \dot{\mathcal{P}})\,\mathcal{P}^{-1} \tag{2.21b}$$

$$\bar{\mathcal{B}} = \mathcal{P}\mathcal{B} \tag{2.21c}$$

$$\bar{\mathcal{C}} = \mathcal{C}\,\mathcal{P}^{-1} \tag{2.21d}$$

where

$$\mathcal{P} = \begin{bmatrix} \mu & O_n \\ O_n & 1_n \end{bmatrix} \tag{2.21e}$$

Equations (2.21) have the same form as the transformation equations for linear time-variable systems [5], but here the transformation matrix $\mathcal{P}$ depends on the state variable [ actually, $\mathcal{P}$ only depends on the output variable, q ] .

Since $\alpha$ and $\nu$ are $n \times n$ nonsingular matrices, all the coefficient matrices ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\bar{\mathcal{A}}$, $\bar{\mathcal{B}}$, $\bar{\mathcal{C}}$ ) have rank n. In fact, these coefficient matrices are analytic functions of the state variables.

The next section discusses equations (2.17) to (2.20) from a system theory point of view.

## III. APPLICABLE SYSTEM THEORY

The previous section provides the working knowledge necessary to relate system theory concepts to spacecraft attitude dynamics problems. In particular, the concepts of controllability, observability, and other algebraic system theory concepts can now be meaningfully applied. In addition, the equations of motion are now in a form such that the terms linear in the state variables can be conveniently separated from the terms which are quadratic and higher.

Equations (2.18a) and (2.20a) show that the general spacecraft attitude dynamics plant has a system state equation of the following specialized form

$$\dot{x} = \mathcal{A}(x,t)x + \mathcal{B}(x,t)u \tag{3.1}$$

where $\mathcal{A}$ and $\mathcal{B}$ are bounded for bounded x. Furthermore, $\mathcal{A}$ and $\mathcal{B}$ are actually analytic in x [except at various discrete times when the order or the structure of the equations might change ]. Equation (3.1) is linear in the control u (=K); this can be emphasized by writing

$$\dot{x} = F(x,t) + \mathcal{B}(x,t)u \tag{3.2}$$

where

$$F(x,t) = \mathcal{A}(x,t)x \tag{3.3}$$

with

$$F(0,t) = 0 \tag{3.4}$$

Equations of the form of (3.2) have been investigated in [16] - [20].

Equations of the form in (3.1) have been studied by Davison and Kunze [21] in terms of the related time-varying linear system defined by

$$\dot{x} = \mathcal{A}(z(t), t)\, x + \mathcal{B}(z(t), t)\, u \tag{3.5}$$

where z(t) is a prescribed function from the set of possible solutions to Equations (3.3) and (3.5).

A. Controllability

Theorem 1 of [21] deals with global controllability and requires that the elements of $\mathcal{A}$ and $\mathcal{B}$ be bounded for all possible z(t). But the $\mathcal{B}$ of equation (2.20a) is obviously bounded since it is a constant. The $\mathcal{B}$ of equation (2.18a) is also bounded since $\nu$ is bounded in multi-body dynamics problems; in fact, the elements of $\nu$ are bounded functions of coordinates which are themselves bounded. However, neither the $\mathcal{A}$ of equation (2.18a) nor the $\mathcal{A}$ of (2.20a) are bounded since each depends on $\gamma$ which is linear in g or G (and nonlinear in q).

Theorem 2 of [21] deals with local controllability which is obtained when $\mathcal{A}$ and $\mathcal{B}$ are bounded over some bounded set of solutions z(t). Controllability of the nonlinear system (3.1) is reduced to controllability of the related linear time-varying system (3.5). But controllability of linear time-varying systems is easily checked via the Silverman-Meadows controllability matrices [22] - [24]

$$\underline{\mathcal{B}}_j = [\, \mathcal{B}^{[0]} \;\; \mathcal{B}^{[1]} \cdots \mathcal{B}^{[j-1]} \,] \tag{3.6a}$$

where

$$\mathcal{B}^{[0]} = \mathcal{B} \tag{3.6b}$$

$$\mathcal{B}^{[1]} = \dot{\mathcal{B}} - \mathcal{A}\mathcal{B} \tag{3.6c}$$

$$\mathcal{B}^{[m+1]} = \left(\frac{d}{dt} - \mathcal{A}\right)\mathcal{B}^{[m]}, \quad m = 0, 1, \ldots 2n-2 \tag{3.6d}$$

For a system of order 2n:

$$\underline{\mathcal{B}} \equiv \underline{\mathcal{B}}_{2n}$$

If i is the lowest index for which rank $\underline{\mathcal{B}}_i$ = rank $\underline{\mathcal{B}}_{i+1}$ = rank $\underline{\mathcal{B}}$, then i is called the controllability index [24]. If rank $\underline{\mathcal{B}}_i = 2n$ (for a system of order 2n), then the system is controllable.

Velocity Formulation

In the velocity formulation, Equations (2.17a) and (2.18a) yield

$$\mathcal{A} = \begin{bmatrix} -\gamma^t & O_n \\ \alpha & O_n \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} \nu \\ O_n \end{bmatrix} \tag{3.7}$$

Therefore, from equations (2.10b) and (3.7)

$$\mathcal{B}^{[1]} = \dot{\mathcal{B}} - \mathcal{A}\mathcal{B} = \begin{bmatrix} -\gamma^t\nu - \nu\gamma \\ O_n \end{bmatrix} - \begin{bmatrix} -\gamma^t\nu \\ \alpha\nu \end{bmatrix} = \begin{bmatrix} -\nu\gamma \\ -\bar{\alpha} \end{bmatrix} \tag{3.8}$$

It follows that

$$\underline{\mathcal{B}}_2 = \begin{bmatrix} \mathcal{B} & \mathcal{B}^{[1]} \end{bmatrix} = \begin{bmatrix} \nu & -\nu\gamma \\ O_n & -\bar{\alpha} \end{bmatrix} \tag{3.9}$$

Since $\nu$ and $\bar{\alpha}$ are nonsingular, it follows that $\underline{\mathcal{B}}_2$ has rank 2n, and therefore the related linear system is controllable, and the controllability index is 2.

Momentum Formulation

In the momentum formulation, equations(2.19a) and (2.20a) yield

$$\bar{\mathcal{A}} = \begin{bmatrix} \gamma & O_n \\ \bar{\alpha} & O_n \end{bmatrix}, \qquad \bar{\mathcal{B}} = \begin{bmatrix} 1_n \\ O_n \end{bmatrix} \tag{3.10}$$

Since $\bar{\mathcal{B}}$ is constant, it follows that

$$\bar{\mathcal{B}}^{[1]} = -\bar{\mathcal{A}}\bar{\mathcal{B}} = -\begin{bmatrix} \gamma \\ \bar{\alpha} \end{bmatrix} \tag{3.11}$$

Therefore

$$\underline{\bar{\mathcal{B}}}_2 = \left[ \bar{\mathcal{B}} \;\; \bar{\mathcal{B}}^{[1]} \right] = \begin{bmatrix} 1_n & -\gamma \\ O_n & -\bar{\alpha} \end{bmatrix} \tag{3.12}$$

Since $\bar{\alpha}$ is nonsingular, it follows that $\underline{\bar{\mathcal{B}}}_2$ has rank 2n, and the momentum formulation is also controllable (as expected). Note that $\underline{\mathcal{B}}_2$ and $\underline{\bar{\mathcal{B}}}_2$ are related by

$$\underline{\bar{\mathcal{B}}}_2 = \mathcal{P}\underline{\mathcal{B}}_2 \tag{3.13}$$

as they should [23]; here $\mathcal{P}$ is given by equation (2.21e).

B. Observability

Observability of the related time-varying linear system can be checked similarly via the Silverman-Meadows observability matrices [22] - [24]

$$\underline{\mathcal{C}}_j = \begin{bmatrix} \mathcal{C}^{[0]} \\ \mathcal{C}^{[1]} \\ \vdots \\ \mathcal{C}^{[j-1]} \end{bmatrix} \tag{3.14a}$$

where

$$\mathcal{C}^{[0]} = \mathcal{C} \tag{3.14b}$$

$$\mathcal{C}^{[1]} = \dot{\mathcal{C}} + \mathcal{C}\mathcal{A} \Longrightarrow \mathcal{C}^{[1]^t} = \dot{\mathcal{C}}^t + \mathcal{A}^t \mathcal{C}^t \tag{3.14c}$$

$$\mathcal{C}^{[m+1]^t} = \left(\frac{d}{dt} + \mathcal{A}^t\right) \mathcal{C}^{[m]^t}, \quad m = 0, 1, \ldots, 2n-2 \tag{3.14d}$$

For a system of order 2n:

$$\underline{\mathcal{C}} \equiv \underline{\mathcal{C}}_{2n} \tag{3.14e}$$

If i is the lowest index for which rank $\underline{\mathcal{C}}_i$ = rank $\underline{\mathcal{C}}_{i+1}$ = rank $\underline{\mathcal{C}}$, then i is called the observability index [24]. If rank $\underline{\mathcal{C}}_i = 2n$ (for a system of order 2n), then the system is observable.

For both the velocity and the momentum formulation, the output is q and therefore

$$\mathcal{C} = [\, O_n \quad 1_n \,] = \bar{\mathcal{C}} \tag{3.15}$$

This matrix is constant and of rank n. $\mathcal{C}\mathcal{A}$ and $\bar{\mathcal{C}}\bar{\mathcal{A}}$ are also of rank n, and the observability index is 2. For the velocity formulation

$$\underline{\mathcal{C}}_2 = \begin{bmatrix} \mathcal{C} \\ \mathcal{C}^{[1]} \end{bmatrix} = \begin{bmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \end{bmatrix} = \begin{bmatrix} O_n & 1_n \\ \alpha & O_n \end{bmatrix} \tag{3.16}$$

For the momentum formulation

$$\underline{\bar{\mathcal{C}}}_2 = \begin{bmatrix} \bar{\mathcal{C}} \\ \bar{\mathcal{C}}^{[1]} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{C}} \\ \bar{\mathcal{C}}\bar{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} O_n & 1_n \\ \bar{\alpha} & O_n \end{bmatrix} \tag{3.17}$$

Evidently, both $\underline{\mathcal{C}}_2$ and $\underline{\bar{\mathcal{C}}}_2$ have rank 2n, and they are related by

$$\underline{\bar{\mathcal{C}}}_2 = \underline{\mathcal{C}}_2 \, \mathcal{P}^{-1} \tag{3.18}$$

as they should [23].

## C. Covariance, Contravariance, and Duality

The duality between controllability and observability is similar to the duality between covariant and contravariant tensors. This conclusion has been stated recently by Hermann and Krener in the wider context of nonlinear differentiable systems [25]. This duality will be illustrated here only for time-variable linear systems.

Consider the linear time-varying system given by

$$\dot{x} = \mathcal{A}(t)\, x + \mathcal{B}(t)\, u \tag{3.19a}$$

$$y = \mathcal{C}(t)\, x \tag{3.19b}$$

Let $\Phi(t, t_0)$ be the transition matrix so that given the state at $t_0$, $x(t_0)$, the state at t is given as

$$x(t) = \Phi(t, t_0)\left[x(t_0) + \int_{t_0}^{t} \Phi(t_0, \tau)\, \mathcal{B}(\tau)\, u(\tau)\, d\tau\right] \tag{3.20}$$

The system is controllable if and only if the rows of $\Phi(t_0, \cdot)\,\mathcal{B}(\cdot)$ are linearly independent functions. This condition is equivalent to the nonsingularity of the controllability Gramian matrix [24]

$$M(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)\, \mathcal{B}(\tau)\, \mathcal{B}^t(\tau)\, \Phi^t(t_0, \tau)\, d\tau \tag{3.21}$$

The relationship between covariance and controllability is now brought out by the fact that $M(t_0, t_1)$ satisfies the linear matrix differential equation [5]

$$\frac{d}{dt} M(t, t_1) = \mathcal{A}(t) M(t, t_1) + M(t, t_1)\, \mathcal{A}^t(t) - \mathcal{B}(t)\, \mathcal{B}^t(t) \tag{3.22}$$

Ignoring $\mathcal{B}$, this equation is similar to equation (2.10a) with M and $\mathcal{A}$ taking the roles of $\mu$ and $\gamma$, respectively.

The system is observable if and only if the columns of $\mathcal{C}(\cdot)\,\Phi(\cdot, t_0)$ are linearly independent functions. This condition is equivalent to the nonsingularity of the observability Gramian matrix [24]

$$N(t_0, t_1) = \int_{t_0}^{t_1} \Phi^t(\tau, t_0)\, \mathcal{C}^t(\tau)\, \mathcal{C}(\tau)\, \Phi(\tau, t_0)\, d\tau \tag{3.23}$$

The relationship between contravariance and observability is now brought out by the fact that $N(t_0, t_1)$ satisfies the linear matrix equation [5]

$$\frac{d}{dt} N(t, t_1) = -\mathcal{A}^t(t) N(t, t_1) - N(t, t_1)\mathcal{A}(t) - \mathcal{C}^t(t)\,\mathcal{C}(t) \tag{3.24}$$

Ignoring $\mathcal{C}$, this equation is similar to equation (2.10b) with $\nu$ and $\gamma$ replacing N and $\mathcal{A}$, respectively.

D. Transformation of Input

Despite the duality between controllability and observability, it appears that there is a lack symmetry because $\gamma$ enters in $\underline{\mathcal{B}}_2$ and $\underline{\bar{\mathcal{B}}}_2$ (equations (3.9) and (3.12), respectively), whereas $\gamma$ does not enter into $\underline{\mathcal{C}}_2$ and $\underline{\bar{\mathcal{C}}}_2$ (equations (3.16) and (3.17), respectively. This dependence on $\gamma$ can be formally eliminated by redefining the input as follows:

$$k' = k - \gamma^t g = \nu(K - \mu\,\gamma^t g) \tag{3.25a}$$

and

$$K' = K + \gamma G \tag{3.25b}$$

Equation (2.17a) then becomes

$$\begin{bmatrix} \dot{g} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} O_n & O_n \\ \alpha & O_n \end{bmatrix} \begin{bmatrix} g \\ q \end{bmatrix} + \begin{bmatrix} 1_n \\ O_n \end{bmatrix} k' \tag{3.26}$$

In effect, the $\gamma$ in $\mathcal{A}$ has been set to zero, and hence the $\gamma$ in $\underline{\mathcal{B}}_2$ is also zero. Of course, the rank of $\underline{\mathcal{B}}_2$ is still 2n, independent of $\gamma$.

Similarly, equation (2.19a) becomes

$$\begin{bmatrix} \dot{G} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} O_n & O_n \\ \bar{\alpha} & O_n \end{bmatrix} \begin{bmatrix} G \\ q \end{bmatrix} + \begin{bmatrix} 1_n \\ O_n \end{bmatrix} K' \tag{3.27}$$

Again, in effect the $\gamma$ in $\bar{\mathcal{A}}$ has been set to zero; this then results in the $\gamma$ in $\bar{\underline{\mathcal{B}}}_2$ being set to zero, with rank of $\bar{\underline{\mathcal{B}}}_2$ remaining 2n.

Redefining the input variable has another important effect: $\mathcal{A}$ and $\bar{\mathcal{A}}$ are now bounded (since the unboundedness was due to $\gamma$). Consequently, by Theorem 1 of [21] the nonlinear systems (3.26) and (3.27) are globally controllable.

## E. Perturbed Linearized System

Equations (2.17a) and (2.19a) are actually even controllable when they are linearized. Linearization involves dropping the term in $\gamma$ and evaluating $\alpha$, $\bar{\alpha}$, and $\nu$ at $q = 0$. More generally, the equations can be written as these linearized terms, plus higher order "perturbations". Thus, equation (2-17a) becomes

$$\begin{bmatrix} \dot{g} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} O_n & O_n \\ \alpha(0) & O_n \end{bmatrix} \begin{bmatrix} g \\ q \end{bmatrix} + \begin{bmatrix} \nu(0) \\ O_n \end{bmatrix} K$$

$$+ \begin{bmatrix} -\gamma^t & O_n \\ \alpha - \alpha(0) & O_n \end{bmatrix} \begin{bmatrix} g \\ q \end{bmatrix} + \begin{bmatrix} \nu - \nu(0) \\ O_n \end{bmatrix} K \tag{3.28}$$

This equation has the general form

$$\dot{x} = \mathcal{A}x + \mathcal{B}u + h(x, u) \tag{3.29}$$

with $\mathcal{A}$ and $\mathcal{B}$ being constants; h can be further expanded as

$$h(x, u) = h_2(x) + h_1(x)u$$

where $h_2$ is quadratic (or higher) in x and $h_1$ is linear (or higher) in x.

Similarly, equation (2.19a) becomes

$$\begin{bmatrix} \dot{G} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} O_n & O_n \\ \bar{\alpha}(0) & O_n \end{bmatrix} \begin{bmatrix} G \\ q \end{bmatrix} + \begin{bmatrix} 1_n \\ O_n \end{bmatrix} K$$

$$+ \begin{bmatrix} \gamma & O_n \\ \bar{\alpha} - \bar{\alpha}(0) & O_n \end{bmatrix} \begin{bmatrix} G \\ q \end{bmatrix} \tag{3.30}$$

This equation has the general form

$$\dot{\bar{x}} = \bar{\mathcal{A}}\bar{x} + \bar{\mathcal{B}}u + \bar{h}(x) \tag{3.31}$$

with $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ being constants, and with $\bar{h}$ being quadratic in $\bar{x}$.

It is evident that the linearized parts of (3.29) and (3.31) are controllable. Systems of this type are studied in [26] - [27]. A slightly more general perturbed system is studied in [28]. Specific studies of quadratic systems are in [29] - [33].

## F. Nonlinear Algebraic System Theory

Loosely put, algebraic system theory is the theory of systems in terms of the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ rather than in terms of the solutions of the system equations. In particular, the state transition matrix does not appear explicitly in this theory. Much of the initial work on algebraic system theory is due to Kalman and coworkers [34] - [35]. The work of Silverman and Meadows [22] is also part of algebraic system theory (for time-variable linear systems) since it deals with controllability and observability in terms of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ rather than in terms of the state transition matrix.

Algebraic system theory is presently still only a linear theory, but it is conjectured that many of the results which depend only on $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ can be extended to nonlinear systems like equation (3.3). For example, the theory of "linear algebra with continuously parametrized elements" in [36] - [37] can probably be extended to a theory of "linear algebra with continuous state-dependent elements". It is conjectured that such a theory could prove that the $\mathcal{A}$ of equation (3.7) can be transformed so that $\gamma$ becomes zero:

$$\mathcal{A} = \begin{bmatrix} -\gamma^t & O_n \\ \alpha & O_n \end{bmatrix} \longrightarrow \begin{bmatrix} O_n & O_n \\ \hat{\alpha} & O_n \end{bmatrix} \tag{3.32}$$

where $\hat{\alpha}$ is a nonsingular $n \times n$ matrix, so that the rank of the new $\mathcal{A}$ is still n. The $\bar{\mathcal{A}}$ of equation (3.10) can be transformed similarly:

$$\bar{\mathcal{A}} = \begin{bmatrix} \gamma & O_n \\ \bar{\alpha} & O_n \end{bmatrix} \longrightarrow \begin{bmatrix} O_n & O_n \\ \hat{\bar{\alpha}} & O_n \end{bmatrix} \tag{3.33}$$

where $\hat{\hat{\alpha}}$ is also a nonsingular $n \times n$ matrix so that the rank of of the new $\bar{\mathcal{A}}$ is still n.

It is not really clear whether or not a change of variables can be found so that $\gamma$ is transformed to zero. Even if such a transformation does exists, it may not be directly useable since it may depend on knowledge of the solution of the differential equations. However, it follows from general tensorial considerations that the form of $\gamma$ can be simplified.

Recall that $\gamma$ depends nonlinearly on q and linearly on g: $\gamma = \gamma(q, g)$. If the matrix elements of $\gamma$ are linearly expanded in terms of the elements of g, then the expansion coefficients are the components of the affine connection (for the chosen basis)[13]; if the elements of g are holonomic velocity components then these coefficients reduce to the Christoffel symbols of the second kind for the metric matrix $\mu$ [11]. But it is known that transforming to geodesic coordinates (or normal coordinates) [11] at the point $p$ yields Christoffel symbols which vanish at the point $p$. If q represents the coordinates of the point $p$, then it follows that $\gamma(0, g) = 0$; thus, $\gamma$ is linear (or higher) in q, and as a result, $\gamma$ is quadratic (or higher) in the state.

Another interesting fact about $\gamma$ is that it can be transformed so as to become skew-symmetric. This is evident on examining equations (2.1a) and (2.3a), and then transforming these equations to $\dot{\tilde{G}} = \tilde{\gamma}\tilde{G} + \tilde{K}$ and $\dot{\tilde{g}} = -\tilde{\gamma}^t\tilde{g} + \tilde{k}$, respectively. If these transformations are made such that $\tilde{G} = \tilde{g}$ (i.e., momentum and velocity are "normalized" by the square-root of the mass) then it follows that $\tilde{\gamma} = -\tilde{\gamma}^t$, which is the condition for skew-symmetry.

In order to be sufficiently general, algebraic system theory must be formulated in terms of manifolds rather than vector spaces because the configuration space for a dynamical system is a manifold, $\mathcal{M}$, and the state space is the tangent bundle, $\mathcal{T}(\mathcal{M})$, of this manifold. An interesting introduction to differentiable manifolds in nonlinear

system theory is the review by Brockett [38]. More advanced material is in [25] and [39] - [40].

The next section casts multi-body dynamics in a nonlinear algebraic system theory setting. The multi-body system is considered to consist of N interconnected subsystems, and the nonlinear equations of motion for the system are algebraically related to the nonlinear equations of motion for the N subsystems. The interconnection constraints among the N subsystems are linear. Therefore, these constraints decompose the (linear) velocity and momentum spaces into "free" and "constrained" parts; these are, in effect, the controllable and uncontrollable subspaces. "Short exact sequences" and "algebraic network diagrams" are used to relate the various subspaces.

## IV. MULTI-BODY DYNAMICS

Consider a (dynamical) system which can be conveniently considered as a collection of N interconnected subsystems. For example, in classical dynamics it is well known [15] that any mechanical system can be considered as a collection of N particles. Each particle has three (translational) degrees of freedom, so that the system has a total of 3N degrees of freedom if the interconnections do not impose any constraints. If the interconnections impose m constraints, then there are $n = 3N - m$ degrees of freedom; the resulting dynamics plant is of order 2n, having n free velocity components and n free position coordinates.

Often a mechanical system can be considered as a collection of N rigid bodies [1] - [4]; each rigid body has six degrees of freedom (three translational and three rotational), so that the system has 6N degrees of freedom before the application of constraints. After the incorporation of m constraints, there are $n = 6N - m$ degrees of freedom; the resulting plant is, of course, again of order 2n. More generally, a mechanical system can be considered as a collection of N deformable bodies, with Body i having $n_{(i)} \geq 1$ degrees of freedom.

### A. Primitive Composite System

Now consider a collection of N bodies, with Body i having $n_{(i)}$ degrees of freedom. The momentum formulations dynamics equations for Body i are

$$\dot{G}_{(i)} = \gamma_{(i)} G_{(i)} + K_{(i)} \tag{4.1a}$$

$$g_{(i)} = \nu_{(i)} G_{(i)} \tag{4.1b}$$



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Here $K_{(i)}$, $G_{(i)}$, and $g_{(i)}$ are column matrices with $n_{(i)}$ elements, and $\gamma_{(i)}$ and $\nu_{(i)}$ are $n_{(i)} \times n_{(i)}$ matrices. The velocity formulation dynamics equations for Body i are

$$\dot{g}_{(i)} = -\gamma^{t}_{(i)} g_{(i)} + \nu_{(i)} K_{(i)} \tag{4.2a}$$

$$g_{(i)} = g_{(i)} \tag{4.2b}$$

$K_{(i)}$, $G_{(i)}$, and $g_{(i)}$ are called the Body i force, momentum, and velocity, respectively; $\nu_{(i)}$ and $\gamma_{(i)}$ are the Body i inverse mass matrix and Cartan matrix, respectively.

Ignoring the constraints, the system force, momentum and velocity are given in terms of direct sums of the corresponding subsystem quantities, as follows:

$$K = \begin{bmatrix} K_{(1)} \\ K_{(2)} \\ \vdots \\ K_{(N)} \end{bmatrix}, \quad G = \begin{bmatrix} G_{(1)} \\ G_{(2)} \\ \vdots \\ G_{(N)} \end{bmatrix}, \quad g = \begin{bmatrix} g_{(1)} \\ g_{(2)} \\ \vdots \\ g_{(N)} \end{bmatrix} \tag{4.3a, b, c}$$

Similarly, the system inverse mass matrix and Cartan matrix are given in terms of direct sums as follows

$$\nu = \begin{bmatrix} \nu_{(1)} & & & \\ & \nu_{(2)} & & \\ & & \ddots & \\ & & & \nu_{(N)} \end{bmatrix} \tag{4.3d}$$

$$\gamma = \begin{bmatrix} \gamma_{(1)} & & & \\ & \gamma_{(2)} & & \\ & & \ddots & \\ & & & \gamma_{(N)} \end{bmatrix} \tag{4.3e}$$

Note that $\nu$ and $\gamma$ are "block-diagonal", with the off-diagonal blocks being zero.

When the constraints are ignored, the momentum formulation system equations of motion are given by

$$\dot{G} = \gamma G + K \tag{4.4a}$$

$$g = \nu G \tag{4.4b}$$

and the velocity formulation system equations are

$$\dot{g} = -\gamma^t g + \nu K \tag{4.5a}$$

$$g = g \tag{4.5b}$$

These equations are, of course, exactly the same as equations (2.1) and (2.3).

The system mass matrix is the direct sum of the subsystem mass matrices:

$$\mu = \begin{bmatrix} \mu_{(1)} & & & \\ & \mu_{(2)} & & \\ & & \ddots & \\ & & & \mu_{(N)} \end{bmatrix} \tag{4.6}$$

$\mu$ is the inverse of $\nu$, and hence

$$G = \mu g \tag{4.7}$$

as in equation (2.2).

Let $T_{(i)}$ denote the kinetic energy of Subsystem i. Then $T_{(i)}$ and $\dot{T}_{(i)}$ satisfy

$$T_{(i)} = \frac{1}{2} G^t_{(i)} g_{(i)} = \frac{1}{2} g^t_{(i)} \mu_{(i)} g_{(i)} = \frac{1}{2} G^t_{(i)} \nu_{(i)} G_{(i)} \tag{4.8a}$$

$$\dot{T}_{(i)} = K^t_{(i)} g_{(i)} \tag{4.8b}$$

The kinetic energy of the system, with the constraints ignored, is then defined by

$$T = \sum_{i=1}^{N} T_{(i)} \tag{4.9}$$

Thus T and $\dot{T}$ satisfy

$$T = \frac{1}{2} G^t g = \frac{1}{2} g^t \mu g = \frac{1}{2} G^t \nu G \tag{4.10a}$$

$$\dot{T} = K^t g \tag{4.10b}$$

as in equations (2.6) and (2.9).

When the constraints are ignored, the system is called the primitive composite system since this system is in fact a collection of N independent and non-interacting subsystems. The primitive composite system is also called the "torn system", after Kron [41].

The primitive composite system equations are converted to the actual or constrained composite system equations by incorporation of the constraint equations. But before doing this, it is of interest to examine how the equations transform under a linear velocity (or force) transformation. This velocity transformation can then be selected to impose the required constraints.

## B. Linear Velocity Transformation

Consider the nonsingular linear velocity transformation

$$g = A\bar{g} \tag{4.11}$$

where $A = A(q)$ depends on the position, but not on the velocity. Let $B$ be the inverse of $A$:

$$B = A^{-1} \tag{4.12}$$

Equation (4.11) can now be inverted as follows:

$$\bar{g} = B\,g \tag{4.13}$$

The velocity transformation $A$ induces momentum and force transformations by $A^t$ as follows:

$$\bar{G} = A^t G \tag{4.14}$$

and

$$\bar{K} = A^t K \tag{4.15}$$

These are the required definitions for $\bar{G}$ and $\bar{K}$ in order for $T$ and $\dot{T}$ to have the same form in terms of both the barred and unbarred quantities:

$$T = \frac{1}{2} G^t g = \frac{1}{2} \bar{G}^t \bar{g} \tag{4.16}$$

$$\dot{T} = K^t g = \bar{K}^t \bar{g} \tag{4.17}$$

In order to get

$$T = \frac{1}{2} g^t \mu g = \frac{1}{2} \bar{g}^t \bar{\mu} \bar{g} \tag{4.18}$$

it is necessary to define

$$\bar{\mu} = A^t \mu A \tag{4.19}$$

There now results the relationship

$$\bar{G} = \bar{\mu} \bar{g} \tag{4.20}$$

which has the same form as $G = \mu g$.

Equations (4.14) to (4.20) also have their inverse counterparts. Thus,

$$G = B^t \bar{G} \tag{4.21}$$

$$K = B^t \bar{K} \tag{4.22}$$

and

$$T = \frac{1}{2} G^t \nu G = \frac{1}{2} \bar{G} \bar{\nu} \bar{G} \tag{4.23}$$

where

$$\bar{\nu} = B \nu B^t = \bar{\mu}^{-1} \tag{4.24}$$

Equation (4.20) can now be inverted as

$$\bar{g} = \bar{\nu} \bar{G} \tag{4.25}$$

which has the same form as $g = \nu G$.

The barred momentum formulation equation of motion is now given by

$$\dot{\overline{G}} = \overline{\gamma}\,\overline{G} + \overline{K} \tag{4.26a}$$

where

$$\overline{\gamma} = A^t \gamma B^t + \dot{A}^t B^t \tag{4.26b}$$

The barred velocity formulation equation of motion is now given by

$$\dot{\overline{g}} = -\overline{\gamma}^t \overline{g} + \overline{k} \tag{4.27a}$$

where

$$\overline{k} = Bk \tag{4.27b}$$

and

$$\overline{\gamma}^t = B\gamma^t A - \dot{B}A \tag{4.27c}$$

Equation (4.27c) is equivalent to (4.26b) since

$$B\,A = 1_n \Longrightarrow \dot{B}\,A = -B\,\dot{A} \tag{4.28a, b}$$

Equation (4.27b) also has the inverse form

$$k = A\overline{k} \tag{4.29}$$

which is analogous to $g = A\,\overline{g}$. From $\overline{K} = A^t K$ and $K = \mu k$ there now follows

$$\overline{K} = \overline{\mu}\,\overline{k} \Longrightarrow \overline{k} = \overline{\nu}\,\overline{K} \tag{4.30a, b}$$

It is interesting to note that all the transformation equations have the same form as would be obtained if A (and B) were time-dependent rather than position dependent. Thus, the transformation theory of linear systems carries over more or less directly to

nonlinear systems with linear velocity (or momentum) transformations. It should also be noted that the coordinate transformation (implied by the above velocity transformation) is not shown; this coordinate transformation is in general nonlinear since the configuration manifold is a nonlinear space. But the associated velocity and momentum transformations are linear since the tangent and cotangent spaces at a point $p$ are linear vector spaces.

## C. Partitions and Decompositions

Generally A and $\bar{g}$ are chosen so that the resulting $\bar{g}$ can be partitioned into a part $\bar{g}_1$ which is "free" or "unknown" and a part $\bar{g}_2$ which is "constrained" or "known":

$$\bar{g} = \begin{bmatrix} \bar{g}_1 \\ \\ \bar{g}_2 \end{bmatrix} \tag{4.31}$$

This partitioning of $\bar{g}$ induces a corresponding partitioning of A:

$$A = [\ A_1 \quad A_2\ ] \tag{4.32}$$

so that the equation $g = A\,\bar{g}$ becomes

$$g = [\ A_1 \quad A_2\ ] \begin{bmatrix} \bar{g}_1 \\ \\ \bar{g}_2 \end{bmatrix} = A_1\bar{g}_1 + A_2\bar{g}_2 \tag{4.33}$$

Now suppose that $\bar{g}_1$ has $n_1$ elements, $\bar{g}_2$ has $n_2$ elements, and $\bar{g}$ has $n = n_1 + n_2$ elements. Then $A_i$ is an $n \times n_i$ matrix with rank $n_i$, for $i = 1, 2$.

The partitioning of $\bar{g}$ also induces a corresponding partitioning of B:

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{4.34}$$

so that the equation $\bar{g} = B\,g$ becomes

$$\begin{bmatrix} \bar{g}_1 \\ \bar{g}_2 \end{bmatrix} = \begin{bmatrix} B_1\,g \\ B_2\,g \end{bmatrix} \tag{4.35}$$

Since A and B are inverses of each other, it follows that

$$A\,B = A_1 B_1 + A_2\,B_2 = 1 \tag{4.36}$$

where 1 is the $n \times n$ identity. Also

$$B\,A = \begin{bmatrix} B_1\,A_1 & B_1\,A_2 \\ B_2\,A_1 & B_2\,A_2 \end{bmatrix} = \begin{bmatrix} 1_1 & O \\ O & 1_2 \end{bmatrix} \tag{4.37}$$

where $1_i$ is the $n_i \times n_i$ identity. Thus $B_i$ is a left inverse of $A_i$, and $A_i$ is a right inverse of $B_i$, for $i = 1, 2$.

Since $\bar{g}_1$ is free whereas $\bar{g}_2$ is constrained, equation (4.33) is a unique decomposition of g into its free and constrained parts. Define $g^1$ and $g^2$ by

$$g^i = A_i\,\bar{g}_i\,, \quad i = 1,\ 2 \tag{4.38}$$

Then equation (4.33) becomes

$$g = g^1 + g^2 \tag{4.39}$$

Next, note that

$$B_i g = B_i g^i = \bar{g}_i , \quad i = 1,2 \tag{4.40}$$

The momentum can be partitioned and decomposed analogously. From $\bar{G} = A^t G$ it is evident that $\bar{G}$ is partitioned into $\bar{G}_1$ and $\bar{G}_2$ as follows:

$$\bar{G} = \begin{bmatrix} \bar{G}_1 \\ \\ \bar{G}_2 \end{bmatrix} = \begin{bmatrix} A_1^t G \\ \\ A_2^t G \end{bmatrix} \tag{4.41}$$

From $G = B^t \bar{G}$ there follows

$$G = B_1^t \bar{G}_1 + B_2^t \bar{G}_2 \tag{4.42}$$

Thus, G is decomposed as

$$G = G^1 + G^2 \tag{4.43}$$

where

$$G^i = B_i^t \bar{G}_i , \quad i = 1,2 \tag{4.44}$$

Next, note that

$$A_i^t G = A_i^t G^i = \overline{G}_i , \quad i = 1,2 \tag{4.45}$$

The force can be partitioned and decomposed analogously. From $\overline{K} = A^t K$ it is evident that $\overline{K}$ is partitioned into $\overline{K}_1$ and $\overline{K}_2$ as follows:

$$\overline{K} = \begin{bmatrix} \overline{K}_1 \\ \\ \overline{K}_2 \end{bmatrix} = \begin{bmatrix} A_1^t K \\ \\ A_2^t K \end{bmatrix} \tag{4.46}$$

$\overline{K}_1$ is the "free" or "control" force, whereas $\overline{K}_2$ is the constraint force. From $K = B^t \overline{K}$ there follows

$$K = B_1^t \overline{K}_1 + B_2^t \overline{K}_2 \tag{4.47}$$

Thus, K is decomposed as

$$K = K^1 + K^2 \tag{4.48}$$

where

$$K^i = B_i^t \overline{K}_i , \qquad i = 1,2 \tag{4.49}$$

Next, note that

$$A_i^t K = A_i^t K^i = \overline{K}_i , \quad i = 1,2 \tag{4.50}$$

Note that the sequence of transformations could have been started with selecting A and $\overline{K}$ so that the resulting $\overline{K}_1$ and $\overline{K}_2$ have the above properties. The velocity transformation which specifies $\overline{g}_1$ and $\overline{g}_2$ would then be an induced transformation rather than the primitive transformation.

The kinetic energy T can now be expressed as

$$T = \frac{1}{2} G^t g = \frac{1}{2} (G^{1^t} g^1 + G^{2^t} g^2) \tag{4.51a}$$

$$= \frac{1}{2} (\bar{G}_1^t \bar{g}_1 + \bar{G}_2^t \bar{g}_2) \tag{4.51b}$$

and the time derivative of the kinetic energy becomes

$$\dot{T} = K^t g = K^{1^t} g^1 + K^{2^t} g^2 \tag{4.52a}$$

$$= \bar{K}_1^t \bar{g}_1 + \bar{K}_2^t \bar{g}_2 \tag{4.52b}$$

Note that the "constrained" (or "prescribed") velocity $\bar{g}_2$ contributes to T and $\dot{T}$, unless $\bar{g}_2$ is zero. From (4.52b) it is evident that if $\bar{g}_2$ is non-zero then the constraint force $\bar{K}_2$ performs work and thus affects the system's energy.

D. <u>Induced Subspaces and Short Exact Sequences</u>

Now introduce the notation

$$\pi^i = A_i B_i \ , \quad i = 1, 2 \tag{4.53}$$

From Equation (4.36) it follows that

$$\pi^1 + \pi^2 = 1 \tag{4.54}$$

Since $B_i$ is a left inverse of $A_i$ it follows that $\pi^i$ is idempotent:

$$\pi^i \pi^i = \pi^i \ , \ i = 1, 2 \tag{4.55}$$

Also

$$\pi^1 \pi^2 = O = \pi^2 \pi^1 \tag{4.56}$$

It follows that $\pi^1$ and $\pi^2$ are orthogonal projectors [42]. These projectors decompose the velocity space $\mathcal{V}$ into direct sums $\mathcal{V}^1$ and $\mathcal{V}^2$ with $g \in \mathcal{V}$ and $g^i \in \mathcal{V}^i$ :

$$\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^2 \tag{4.57}$$

where $\oplus$ denotes a direct sum of vector spaces. Similarly, let $\bar{g} \in \bar{\mathcal{V}}$ and $\bar{g}_i \in \bar{\mathcal{V}}_i$ ; then $\bar{\mathcal{V}}$ is the direct sum of $\bar{\mathcal{V}}_1$, and $\bar{\mathcal{V}}_2$ :

$$\bar{\mathcal{V}} = \bar{\mathcal{V}}_1 \oplus \bar{\mathcal{V}}_2 \tag{4.58}$$

Since g and $\bar{g}$ are related by a nonsingular linear transformation, it follows that $\mathcal{V}$ and $\bar{\mathcal{V}}$ are isomorphic:

$$\mathcal{V} \simeq \bar{\mathcal{V}} \tag{4.59}$$

The projectors $\pi^{i^t}$ decompose the dual space $\mathcal{G} = \mathcal{V}^*$ into direct sums $\mathcal{G}^1$ and $\mathcal{G}^2$ with $G \in \mathcal{G}$ and $G^i \in \mathcal{G}^i$ :

$$\mathcal{G} = \mathcal{G}^1 \oplus \mathcal{G}^2 \tag{4.60}$$

Similarly $\bar{\mathcal{G}} = \bar{\mathcal{V}}^*$ is decomposed into direct sums $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$ with $\bar{G} \in \bar{\mathcal{G}}$ and $\bar{G}_i \in \bar{\mathcal{G}}_i$ :

$$\bar{\mathcal{G}} = \bar{\mathcal{G}}_1 \oplus \bar{\mathcal{G}}_2 \tag{4.61}$$

Again, $\mathcal{G}$ and $\bar{\mathcal{G}}$ are isomorphic:

$$\mathcal{G} \simeq \bar{\mathcal{G}} \tag{4.62}$$

The partitions of A and B can be used to form four short exact sequences [43]. Two of these sequences deal with $\bar{\mathcal{V}}_1$ and $\bar{\mathcal{V}}_2$:

$$\bar{\mathcal{V}}_1 \xrightarrow{A_1} \mathcal{V} \xrightarrow{B_2} \bar{\mathcal{V}}_2 \tag{4.63a}$$

$$\bar{\mathcal{V}}_2 \xrightarrow{A_2} \mathcal{V} \xrightarrow{B_1} \bar{\mathcal{V}}_1 \tag{4.63b}$$

Two more sequences deal with $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$:

$$\bar{\mathcal{G}}_1 \xrightarrow{B_1^t} \mathcal{G} \xrightarrow{A_2^t} \bar{\mathcal{G}}_2 \tag{4.64a}$$

$$\bar{\mathcal{G}}_2 \xrightarrow{B_2^t} \mathcal{G} \xrightarrow{A_1^t} \bar{\mathcal{G}}_1 \tag{4.64b}$$

Equations (4.63a) and (4.64b) can be combined as follows

$$\begin{tikzcd}
\bar{\mathcal{V}}_1 \arrow[r, "A_1"] \arrow[d, dashed, "\bar{\mu}_{11}"'] & \mathcal{V} \arrow[r, "B_2"] \arrow[d, shift right, "\mu"'] & \bar{\mathcal{V}}_2 \\
\bar{\mathcal{G}}_1 & \mathcal{G} \arrow[l, "A_1^t"] \arrow[u, shift right, "\nu"'] & \bar{\mathcal{G}}_2 \arrow[l, "B_2^t"] \arrow[u, dashed, "\bar{\nu}_{22}"']
\end{tikzcd} \tag{4.65}$$

Similarly, equations (4.63b) and (4.64a) can be combined as follows:

$$\begin{tikzcd}
\bar{\mathcal{V}}_2 \arrow[r, "A_2"] \arrow[d, dashed, "\bar{\mu}_{22}"'] & \mathcal{V} \arrow[r, "B_1"] \arrow[d, shift right, "\mu"'] & \bar{\mathcal{V}}_1 \\
\bar{\mathcal{G}}_2 & \mathcal{G} \arrow[l, "A_2^t"'] \arrow[u, shift right, "\nu"'] & \bar{\mathcal{G}}_1 \arrow[l, "B_1^t"'] \arrow[u, dashed, "\bar{\nu}_{11}"']
\end{tikzcd} \tag{4.66}$$

These last two diagrams look like the "algebraic network diagrams" of Roth [44] - [45] and Branin [46] - [47]. These diagrams can be filled in as shown via dashed lines. The notation for $\bar{\mu}_{ii}$ and $\bar{\nu}_{ii}$, for $i = 1, 2$, shown in these diagrams is obtained from the partitioned forms of $\bar{\mu}$ and $\bar{\nu}$. Thus

$$\bar{\mu} = A^t \mu A = \begin{bmatrix} A_1^t \mu A_1 & A_1^t \mu A_2 \\ A_2^t \mu A_1 & A_2^t \mu A_2 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\mu}_{11} & \bar{\mu}_{12} \\ \bar{\mu}_{21} & \bar{\mu}_{22} \end{bmatrix} \tag{4.67}$$

Similarly,

$$\bar{\nu} = B \nu B^t = \begin{bmatrix} B_1 \nu B_1^t & B_1 \nu B_2^t \\ B_2 \nu B_1^t & B_2 \nu B_2^t \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\nu}_{11} & \bar{\nu}_{12} \\ \bar{\nu}_{21} & \bar{\nu}_{22} \end{bmatrix} \tag{4.68}$$

Since $\mu$ and $\nu$ are positive definite symmetric, and $A_i$ and $B_i$ have full rank, it follows that $\mu_{ii}$ and $\nu_{ii}$ are also positive definite symmetric, and, therefore, invertible.

From (4.65), for example, it should not be concluded that $\bar{G}_1$ is equal to $\bar{\mu}_{11}\bar{g}_1$ ; this is so only if $\bar{g}_2$ is zero. In general, from $\bar{G} = \bar{\mu}\bar{g}$, or from (4.65) plus (4.66) there follows

$$\bar{G}_1 = \bar{\mu}_{11}\bar{g}_1 + \bar{\mu}_{12}\bar{g}_2 \tag{4.69a}$$

$$\bar{G}_2 = \bar{\mu}_{21}\bar{g}_1 + \bar{\mu}_{22}\bar{g}_2 \tag{4.69b}$$

The inverse relationship, $\bar{g} = \bar{\nu}\bar{G}$ , yields

$$\bar{g}_1 = \bar{\nu}_{11}\bar{G}_1 + \bar{\nu}_{12}\bar{G}_2 \tag{4.70a}$$

$$\bar{g}_2 = \bar{\nu}_{21}\bar{G}_1 + \bar{\nu}_{22}\bar{G}_2 \tag{4.70b}$$

## E. Constrained Motion Equations

Using equation (4.47) for K, the momentum formulation dynamics equation can be written as

$$\dot{G} = \gamma G + B_1^t \bar{K}_1 + B_2^t \bar{K}_2 \tag{4.71}$$

Here $\bar{K}_2$ is the unknown constraint force which can be determined via use of the constraint equation $B_2 g = \bar{g}_2$ . Writing this constraint equation in terms of momentum yields

$$B_2 \nu G = \bar{g}_2 \tag{4.72}$$

Differentiating this equation with respect to time and then combining with equation (4.71) yields

$$\begin{bmatrix} 1_n & -B_2^t \\ B_2 \nu & O \end{bmatrix} \begin{bmatrix} \dot{G} \\ \overline{K}_2 \end{bmatrix} = \begin{bmatrix} \gamma G + B_1^t \overline{K}_1 \\ \dot{\overline{g}}_2 - \dot{B}_2 \nu G - B_2 \dot{\nu} G \end{bmatrix} \tag{4.73}$$

Solving this for $\dot{G}$ yields $\dot{G}$ with $\overline{K}_2$ eliminated (solving the equation, of course, also yields $\overline{K}_2$). However, the resulting equation for $\dot{G}$ is not a minimal dimension realization of the dynamics, and hence the system is not controllable.

To get a minimal dimension realization, left-multiply equation (4.71) by $A_1^t$. Using

$$A_1^t B_1^t = 1_1 , \qquad A_1^t B_2^t = O \tag{4.74a,b}$$

then yields

$$A_1^t \dot{G} = A_1^t \gamma G + \overline{K}_1 \tag{4.75}$$

Thus, $\overline{K}_2$ has been eliminated. Differentiating $\overline{G}_1 = A_1^t G$ with respect to time yields

$$\dot{\overline{G}}_1 = A_1^t \dot{G} + \dot{A}_1^t G \tag{4.76}$$

Combining equations (4.75) and (4.76) yields

$$\dot{\bar{G}}_1 = ( A_1^t \gamma + \dot{A}_1^t ) G + \bar{K}_1 \tag{4.77a}$$

$$= ( A_1^t \gamma + \dot{A}_1^t ) B^t \bar{G} + \bar{K}_1 \tag{4.77b}$$

In this last equation $\bar{G}$ involves both $\bar{G}_1$ and $\bar{G}_2$. $\bar{G}_1$ is, of course, the momentum state variable for the reduced (and controllable) system. $\bar{G}_2$ can be expressed in terms of $\bar{G}_1$ and the prescribed velocity $\bar{g}_2$ as follows. From (4.69a) $\bar{g}_1$ is given as

$$\bar{g}_1 = \bar{\mu}_{11}^{-1} ( \bar{G}_1 - \bar{\mu}_{12} \bar{g}_2 ) \tag{4.78}$$

$\bar{G}_2$ is now obtained by substituting this expression into (4.69b). The result is

$$\bar{G}_2 = \bar{\mu}_{21} \bar{\mu}_{11}^{-1} \bar{G}_1 + ( \bar{\mu}_{22} - \bar{\mu}_{21} \bar{\mu}_{11}^{-1} \bar{\mu}_{12} ) \bar{g}_2 \tag{4.79}$$

If preferred, the use of $\bar{\mu}_{11}^{-1}$ can be replaced with the use of $\bar{\nu}_{22}^{-1}$ as follows

$$\bar{\mu}_{11}^{-1} = \bar{\nu}_{11} - \bar{\nu}_{12} \bar{\nu}_{22}^{-1} \bar{\nu}_{21} \tag{4.80}$$

Note that the coefficient of $\bar{g}_2$ in equation (4.79) is just $\bar{\nu}_{22}^{-1}$:

$$\bar{\nu}_{22}^{-1} = \bar{\mu}_{22} - \bar{\mu}_{21} \bar{\mu}_{11}^{-1} \bar{\mu}_{12} \tag{4.81}$$

Thus, $\bar{G}_2$ can also be expressed as

$$\bar{G}_2 = \bar{\mu}_{21} ( \bar{\nu}_{11} - \bar{\nu}_{12} \bar{\nu}_{22}^{-1} \bar{\nu}_{21} ) \bar{G}_1 + \bar{\nu}_{22}^{-1} \bar{g}_2 \tag{4.82}$$

The constrained motion equations have been given only for the momentum formulation, but the velocity formulation equations are very similar [3]. In general, there is no significant conceptual difference between the momentum formulation and the velocity formulation, and between carrying the constraints as a side condition (as in equation (4.73)) and eliminating the constraints (as in equation (4.77)). Whatever approach is selected, there is the choice of (effectively) inverting $\bar{\mu}_{11}$ or $\bar{\nu}_{22}$, both of which are positive definite and symmetric. However, in particular multi-body systems, one of these four approaches may have computational advantages over the others [3].

The search for various alternative formulations of multi-body dynamics is analogous to the search for various minimal and "canonical" realizations in system theory. Although various realizations may be equivalent, different points of view and different insights can be obtained from different approaches.

## V. DISCUSSION AND CONCLUSION

The previous three sections discuss spacecraft attitude dynamics and multi-body dynamics from the point of view of system theory. It is hoped that this discussion is of interest to system theorist, who may be looking for a sample problem of modest complexity, and to dynamicists, who might be interested in the techniques of system theory.

It is expected that in time the communication gap between system theorists and dynamicists will narrow. This narrowing is expected to occur most rapidly in the field of spacecraft attitude dynamics where the trends are to larger size plants (both mathematically and physically) and more sophisticated controllers (including micro-processors in both the sensors and control laws). For a geometric perspective on the relationship of control theory and mechanics see [48].

The reader interested in more details on multi-body dynamics is referred to [1] - [3] and [49] - [56]. In these references the momentum formulation equation of motion has the general form

$$\dot{G} + X = K \tag{5.1}$$

and the velocity formulation equation of motion is

$$\mu \dot{g} + Y = K \tag{5.2}$$

The relationship between X and Y is

$$Y = X + \dot{\mu} g \tag{5.3}$$



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Comparing Equations (5.1) and (5.2) with Equations (2.1a) and (2.3a), respectively, shows that

$$X = -\gamma G \tag{5.4a}$$

$$Y = \mu \gamma^t g \tag{5.4b}$$

Equations (5.4) are useful in relating the multi-body dynamics formulations in the references with the formulation presented in this paper.

Left-multiplying Equations (5.1) and (5.2) by $\nu = \mu^{-1}$ yields

$$\nu \dot{G} + x = k \tag{5.5}$$

and

$$\dot{g} + y = k \tag{5.6}$$

where

$$x = \nu X = -\nu \gamma G \tag{5.7a}$$

$$y = \nu Y = \gamma^t g \tag{5.7b}$$

$$k = \nu K \tag{5.7c}$$

Of course the variables x and y introduced here differ from the x and y of Sections II and III.

Equations (5.2) and (5.5) are generally not as convenient as Equations (5.6) and (5.1), respectively, but they are given here for completeness. A very practical question is which of these four formulations is computationally most convenient. As pointed out in Section II, generally it is not convenient to find $\gamma$ since $\gamma$ itself is not required in the equations of motion. Similarly, $\dot{\mu}$ and $\dot{\nu}$ are not required since these terms always occur in the combinations of $\dot{\mu} g$ and $\dot{\nu} G$. It is easy to see that

$$Y - X = \dot{\mu} g = (\gamma_G \mu + \mu \gamma_g^t) g \tag{5.8a}$$

$$y - x = -\dot{\nu} G = (\gamma_g^t \nu + \nu \gamma_G) G \tag{5.8b}$$

where $\gamma$ has been decomposed into

$$\gamma = \gamma_G + \gamma'_G \quad \text{where} \quad \gamma'_G G = 0 \tag{5.9a}$$

and

$$\gamma = \gamma_g + \gamma'_g \quad \text{where} \quad \gamma'^t_g g = 0 \tag{5.9b}$$

It now follows that only the $\gamma_G$ part of $\gamma$ is required in the momentum formulation whereas only the $\gamma_g$ part of $\gamma$ is required in the velocity formulation.

Examples which illustrate many of the ideas and concepts in this paper are contained in the Appendix of a forthcoming Aerospace Corporation Technical Report (TR) with the same title as this paper.

The work reported in this paper is a natural extension of the author's earlier work on the transformation operator approach to multi-body dynamics. This earlier work in turn, is an extension or adaptation of some basic ideas in Kron's method of subspaces; see [57] and [58, Part A].

## APPENDIX

The purpose of this appendix is to illustrate some of the key ideas and concepts presented in the paper. Two examples are considered. The first is a single rigid body viewed as a system of N particles. As a byproduct, the Cartan operator for a rigid body is derived.

The second example is the interconnection of two rigid bodies. This illustrates the fact that even though the equations of motion are nonlinear differential equations, nevertheless, the interconnection constraint equations are linear in velocity, momentum, and force.

In the following a classical 3-dimensional vector (or first order tensor) is denoted by an overbar with a single arrowhead; a classical 3-dimensional dyadic (or second order tensor) is denoted by an overbar with a double arrowhead.

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## A. FROM PARTICLES TO A RIGID BODY

A single rigid body has six degrees of freedom. The three rotational degrees of freedom are determined by the rotational equation of motion

$$\dot{\vec{H}}_c = \vec{L}_c \tag{A.1a}$$

where $\vec{L}_c$ is the torque (or moment of force) about the center of mass c, $\vec{H}_c$ is the angular momentum (or moment of linear momentum) about c, and the dot over $\vec{H}_c$ denotes the inertial time derivative. The three translational degrees of freedom are determined by the translational equation of motion

$$\dot{\vec{P}} = \vec{F} \tag{A.1b}$$

where $\vec{F}$ is the force and $\vec{P}$ is the linear momentum. The momenta are linearly related to the velocities as follows:

$$\vec{H}_c = \overleftrightarrow{I} \cdot \vec{\omega} \tag{A.2a}$$

$$\vec{P} = \overleftrightarrow{M} \cdot \vec{v}_c$$

where $\overleftrightarrow{I}$ is the inertia dyadic, $\overleftrightarrow{M}$ is the mass dyadic, $\vec{\omega}$ is the angular velocity, and $\vec{v}_c$ is the linear velocity of the center of mass c. $\overleftrightarrow{I}$ and $\overleftrightarrow{M}$ are positive definite symmetric dyadics which have the positive definite symmetric inverses $\overleftrightarrow{J}$ and $\overleftrightarrow{W}$, respectively:

$$\overleftrightarrow{J} = \overleftrightarrow{I}^{-1} \tag{A.3a}$$

$$\overleftrightarrow{W} = \overleftrightarrow{M}^{-1} \tag{A.3b}$$

Consequently equations (A. 2) have the inverse relationships

$$\vec{\omega} = \overleftrightarrow{J} \cdot \vec{H}_c \tag{A.4a}$$

$$\vec{v}_c = \overleftrightarrow{W} \cdot \vec{P} \tag{A.4b}$$

Equations (A. 1) are the explicit momentum formulation equations of the form $\dot{G} + X = K$. Equations (A. 2) and (A. 4) are the linear relationships $G = \mu \cdot g$ and $g = \nu \cdot G$, respectively.

The coupled velocity formulation equations of the form $\mu \cdot \dot{g} + Y = K$ are given by

$$\vec{I} \cdot \dot{\vec{\omega}} + \tilde{\omega} \cdot \vec{H}_c = \vec{L}_c \tag{A.5a}$$

$$\overleftrightarrow{M} \cdot \dot{\vec{v}}_c = \vec{F} \tag{A.5b}$$

where $\tilde{\omega}$ is the skew-symmetric dyadic of the vector $\vec{\omega}$ (so that $\tilde{\omega} \cdot \vec{H}_c = \vec{\omega} \times \vec{H}_c$).

The mass and inverse mass dyadics are not only positive definite symmetric, but they are also spherical. Thus,

$$\overleftrightarrow{M} = M \overleftrightarrow{E} \tag{A.6a}$$

$$\overleftrightarrow{W} = W \overleftrightarrow{E} \tag{A.6b}$$

$$W = M^{-1} \tag{A.6c}$$

where $M$ and $W$ are the scalar mass and inverse mass, respectively, and $\overleftrightarrow{E}$ is the identity dyadic.

## A.1. System of N Particles

By considering the rigid body to be a system of N particles, the above equations of motion for the rigid body can be derived from the equations of motion for a particle plus the interconnection constraint equations. Let $m_i$ be the mass of the $i^{th}$ particle and let $\vec{r}_i$ be its position vector from an inertial reference origin. The inertial time derivative of $\vec{r}_i$ is the velocity $\vec{v}_i$ ; thus, $\vec{v}_i = \dot{\vec{r}}_i$ .

The center of mass c of the system of N particles has position vector $\vec{r}_c$ where

$$M \vec{r}_c = \sum_{i=1}^{N} m_i \vec{r}_i \tag{A.7a}$$

where

$$M = \sum_{i=1}^{N} m_i \tag{A.7b}$$

Let $\vec{R}_{ic}$ be the position vector to the $i^{th}$ particle from the center of mass. Then $\vec{r}_i$ can be decomposed as follows

$$\vec{r}_i = \vec{R}_{ic} + \vec{r}_c \tag{A.8}$$

Taking the inertial time derivative of this equation yields

$$\vec{v}_i = \dot{\vec{R}}_{ic} + \vec{v}_c \tag{A.9}$$

where $\vec{v}_c = \dot{\vec{r}}_c$ is the velocity of the center of mass which satisfies the relationship

$$M \vec{v}_c = \sum_{i=1}^{N} m_i \vec{v}_i \tag{A.10}$$

The relative position vector $\vec{R}_{ic}$ is fixed in the rigid body. Hence the time variation of $\vec{R}_{ic}$ is due entirely to the rotation with angular velocity $\vec{\omega}$; consequently

$$\begin{aligned} \dot{\vec{R}}_{ic} &= \vec{\omega} \times \vec{R}_{ic} = -\vec{R}_{ic} \times \vec{\omega} \\ &= -\tilde{R}_{ic} \cdot \vec{\omega} = \tilde{R}^t_{ic} \cdot \vec{\omega} \end{aligned} \tag{A.11}$$

where $\tilde{R}^t$ is the dyadic transpose of the dyadic $\tilde{R}$. Since $\tilde{R}$ is skew-symmetric it follows that $\tilde{R}^t = -\tilde{R}$.

Equation (A. 9) can now be written as

$$\vec{v}_i = \tilde{R}^t_{ic} \cdot \vec{\omega} + \vec{v}_c \tag{A.12}$$

From this there follows the matrix-vector-dyadic expression

$$\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_N \end{bmatrix} = \begin{bmatrix} \tilde{R}^t_{1c} & \overleftrightarrow{E} \\ \tilde{R}^t_{2c} & \overleftrightarrow{E} \\ \vdots & \vdots \\ \tilde{R}^t_{Nc} & \overleftrightarrow{E} \end{bmatrix} \cdot \begin{bmatrix} \vec{\omega} \\ \vec{v}_c \end{bmatrix} \tag{A.13}$$

This equation has the general form

$$g = A \cdot \bar{g} \tag{A.14}$$

where $g$ and $\bar{g}$ are column matrices of vectors and A is a matrix of dyadics. The equation $\bar{G} = A^t \cdot G$ takes the form

$$\begin{bmatrix} \vec{H}_c \\ \vec{P} \end{bmatrix} = \begin{bmatrix} \tilde{R}_{1c} & \tilde{R}_{2c} & \cdots & \tilde{R}_{Nc} \\ \overleftrightarrow{E} & \overleftrightarrow{E} & \cdots & \overleftrightarrow{E} \end{bmatrix} \cdot \begin{bmatrix} \vec{p}_1 \\ \vec{p}_2 \\ \vdots \\ \vec{p}_N \end{bmatrix} \tag{A.15}$$

where $\vec{p}_i = m_i \vec{v}_i$ is the linear momentum of the $i^{th}$ particle.

Thus, the equation $\bar{G} = A^t \cdot G$ yields the fundamental results

$$\vec{H}_c = \sum_{i=1}^{N} \tilde{R}_{ic} \cdot \vec{p}_i \tag{A.16a}$$

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i \tag{A.16b}$$

Note that $\vec{H}_c$ and $\vec{P}$ are the new momentum variables associated with the new velocity variables $\vec{\omega}$ and $\vec{v}_c$. Similarly, the equation $\bar{K} = A^t \cdot K$ yields the fundamental results

$$\vec{L}_c = \sum_{i=1}^{N} \tilde{R}_{ic} \cdot \vec{f}_i \tag{A.17a}$$

$$\vec{F} = \sum_{i=1}^{N} \vec{f}_i \tag{A.17b}$$

where $\vec{f}_i$ is the force on the $i^{th}$ particle, so that Newton's Law of Motion for the particle becomes

$$m_i \dot{\vec{v}}_i = \vec{f}_i \tag{A.18}$$

Again, note that $\vec{L}_c$ and $\vec{F}$ are the new force variables associated with the new velocity variables $\vec{\omega}$ and $\vec{v}_c$ .

The equation $G = \mu \cdot g$ takes the form

$$\begin{bmatrix} \vec{p}_1 \\ \vec{p}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{p}_N \end{bmatrix} = \begin{bmatrix} m_1 \overleftrightarrow{E} & & & \\ & m_2 \overleftrightarrow{E} & & \\ & & \ddots & \\ & & & m_N \overleftrightarrow{E} \end{bmatrix} \cdot \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{v}_N \end{bmatrix} \tag{A.19}$$

where the off-diagonal elements of $\mu$ are zero dyadics $\overleftrightarrow{O}$ .

The equation $\bar{\mu} = A^t \cdot \mu \cdot A$ now yields the result

$$\mu = \begin{bmatrix} \sum_{i=1}^{N} m_i \tilde{R}_{ic} \cdot \tilde{R}_{ic}^t & \sum_{i=1}^{N} m_i \tilde{R}_{ic} \\ \sum_{i=1}^{N} m_i \tilde{R}_{ic}^t & \sum_{i=1}^{N} m_i \overleftrightarrow{E} \end{bmatrix} \tag{A.20}$$

$$= \begin{bmatrix} \overleftrightarrow{I} & \overleftrightarrow{O} \\ \overleftrightarrow{O} & \overleftrightarrow{M} \end{bmatrix}$$

Thus, the equation $\bar{\mu} = A^t \cdot \mu \cdot A$ yields the fundamental result

$$\overleftrightarrow{I} = \sum_{i=1}^{N} m_i \tilde{R}_{ic} \cdot \tilde{R}_{ic}^t \tag{A.21}$$

Note that $\overleftrightarrow{I}$ and $\overleftrightarrow{M}$ form the block-diagonal dyadic mass operator associated with the new velocity and momentum variables.

## A.2 Evaluation of Cartan Operator

There remains the evaluation of the new Cartan operator $\bar{\gamma}$ from the expression $\bar{\gamma} = A^t \cdot \gamma \cdot B^t + \dot{A}^t \cdot B^t$. Since $\gamma$ is zero, $\bar{\gamma}$ reduces to

$$\bar{\gamma} = \dot{A}^t \cdot B^t \tag{A.22}$$

A is obtained from Equation (A.13). This operator is one-to-one and therefore has a left inverse (in fact, an infinite number of left inverses). A specific left inverse B is chosen by letting B be the pseudo-inverse of A as follows

$$B = \bar{\nu} \cdot A^t \cdot \mu \tag{A.23}$$

where

$$\bar{\nu} = \begin{bmatrix} \overleftrightarrow{J} & \overleftrightarrow{O} \\ \overleftrightarrow{O} & \overleftrightarrow{W} \end{bmatrix} \tag{A.24}$$

Then

$$\bar{\gamma} = \dot{A}^t \cdot \mu \cdot A \cdot \bar{\nu} \tag{A.25}$$

For convenience, introduce $\bar{\delta}$ as follows

$$\bar{\delta} = \bar{\gamma} \cdot \bar{\mu} = \dot{A}^t \cdot \mu \cdot A \tag{A.26}$$

Performing the indicated operations yields

$$\bar{\delta} = \begin{bmatrix} \sum_{i=1}^{N} m_i \dot{\tilde{R}}_{ic} \cdot \tilde{R}_{ic}^{t} & \sum_{i=1}^{N} m_i \dot{\tilde{R}}_{ic} \\ \overleftrightarrow{O} & \overleftrightarrow{O} \end{bmatrix} \tag{A.27}$$

$$= \begin{bmatrix} \overleftrightarrow{D} & \overleftrightarrow{O} \\ \overleftrightarrow{O} & \overleftrightarrow{O} \end{bmatrix}$$

It now follows that

$$\bar{\gamma} = \bar{\delta} \cdot \bar{\nu} = \begin{bmatrix} \overleftrightarrow{D} \cdot \overleftrightarrow{J} & \overleftrightarrow{O} \\ \overleftrightarrow{O} & \overleftrightarrow{O} \end{bmatrix}$$

$$= \begin{bmatrix} \overleftrightarrow{C} & \overleftrightarrow{O} \\ \overleftrightarrow{O} & \overleftrightarrow{O} \end{bmatrix} \tag{A.28}$$

The evaluation of $\overleftrightarrow{D}$ requires several uses of the identity

$$\widetilde{\vec{A} \times \vec{B}} = \widetilde{\vec{A}} \cdot \vec{B} = \tilde{A} \cdot \tilde{B} - \tilde{B} \cdot \tilde{A} \tag{A.29}$$

Consequently, from equation (A.11) there now follows

$$\dot{\tilde{R}}_{ic} = - \tilde{R}_{ic} \cdot \tilde{\omega} + \tilde{\omega} \cdot \tilde{R}_{ic} \tag{A.30}$$

Hence

$$\overleftrightarrow{D} = \sum_{i=1}^{N} m_i \dot{\tilde{R}}_{ic} \cdot \tilde{R}_{ic}^{t}$$

$$= - \sum_{i=1}^{N} m_i \tilde{R}_{ic} \cdot \tilde{\omega} \cdot \tilde{R}_{ic}^{t} + \sum_{i} m_i \tilde{\omega} \cdot \tilde{R}_{ic} \cdot \tilde{R}_{ic}^{t} \tag{A.31}$$

The second term on the right is just $\tilde{\omega} \cdot \overleftrightarrow{I}$ . The first term on the right can be converted to an expression involving $\tilde{H}_c$ by first writing $\vec{H}_c$ as follows

$$\vec{H}_c = \overleftrightarrow{I} \cdot \vec{\omega} = - \sum_{i=1}^{N} m_i \tilde{R}_{ic} \cdot \tilde{R}_{ic} \cdot \vec{\omega}$$

$$= - \sum_{i=1}^{N} m_i \tilde{R}_{ic} \cdot \vec{V} \tag{A.32}$$

where

$$\vec{V} = \tilde{R}_{ic} \cdot \vec{\omega} \tag{A-33}$$

Hence

$$\tilde{V} = \tilde{R}_{ic} \cdot \tilde{\omega} - \tilde{\omega} \cdot \tilde{R}_{ic} \tag{A.34}$$

It follows that

$$\tilde{H}_c = -\sum_{i=1}^{N} m_i \tilde{R}_{ic} \cdot \tilde{V} + \sum_{i=1}^{N} m_i \tilde{V} \cdot \tilde{R}_{ic}$$

$$= -\sum_{i=1}^{N} m_i \tilde{R}_{ic} \cdot \tilde{R}_{ic} \cdot \tilde{\omega} + \sum_{i=1}^{N} m_i \tilde{R}_{ic} \cdot \tilde{\omega} \cdot \tilde{R}_{ic}$$

$$+ \sum_{i=1}^{N} m_i \tilde{R}_{ic} \cdot \tilde{\omega} \cdot \tilde{R}_{ic} - \sum_{i=1}^{N} m_i \tilde{\omega} \cdot \tilde{R}_{ic} \cdot \tilde{R}_{ic} \tag{A.35}$$

$$= \overleftrightarrow{I} \cdot \tilde{\omega} + 2\sum_{i=1}^{N} m_i \tilde{R}_{ic} \cdot \tilde{\omega} \cdot \tilde{R}_{ic} + \tilde{\omega} \cdot \overleftrightarrow{I}$$

The first term on the right of equation (A. 31) is thus given by

$$\sum_{i=1}^{N} m_i \tilde{R}_{ic} \cdot \tilde{\omega} \cdot \tilde{R}_{ic} = \frac{1}{2} (\tilde{H}_c - \overleftrightarrow{I} \cdot \tilde{\omega} - \tilde{\omega} \cdot \overleftrightarrow{I}) \tag{A.36}$$

Substituting this into $\overleftrightarrow{D}$ now yields

$$\overleftrightarrow{D} = \frac{1}{2} (\tilde{H}_c + \tilde{\omega} \cdot \overleftrightarrow{I} + \overleftrightarrow{I} \cdot \tilde{\omega}^t) \tag{A.37}$$

From $\overleftrightarrow{C} = \overleftrightarrow{D} \cdot \overleftrightarrow{J}$ there now follows

$$\overleftrightarrow{C} = \frac{1}{2} ( \tilde{\omega} + \tilde{H}_c \cdot \overleftrightarrow{J} + \overleftrightarrow{I} \cdot \tilde{\omega}^t \cdot \overleftrightarrow{J} ) \tag{A.38}$$

Since $\overline{\gamma}$ in equation (A.28) is block diagonal it follows that in the equation $\dot{\overline{G}} = \overline{\gamma} \cdot \overline{G} + \overline{K}$ there is no coupling between the rotational and translational equations of motion. Thus, using the $\overline{\gamma}$ from equation (A.28) now leads to the following equations of motion:

$$\dot{\vec{H}}_c = \overleftrightarrow{C} \cdot \vec{H}_c + \vec{L}_c \tag{A.39a}$$

$$\dot{\vec{\omega}} = - \overleftrightarrow{C}^t \cdot \vec{\omega} + \overleftrightarrow{J} \cdot \vec{L}_c \tag{A.39b}$$

Comparing equations (A.39a) and (A.la) yields the requirement

$$\overleftrightarrow{C} \cdot \vec{H}_c = \vec{0} \tag{A.40a}$$

Similarly, comparing equations (A. 39b) and (A. 5a) yields the requirement

$$\overleftrightarrow{I} \cdot \overleftrightarrow{C}^t \cdot \vec{\omega} = \tilde{\omega} \cdot \vec{H}_c = \tilde{H}_c^t \cdot \vec{\omega} \tag{A. 40b}$$

Equations (A. 40a) can easily be verified with the use of $\overleftrightarrow{C}$ in equation (A. 38):

$$\overleftrightarrow{C} \cdot \vec{H}_c = \frac{1}{2} ( \tilde{\omega} \cdot \vec{H}_c + \tilde{H}_c \cdot \overleftrightarrow{J} \cdot \vec{H}_c + \overleftrightarrow{I} \cdot \tilde{\omega}^t \cdot \overleftrightarrow{J} \cdot \vec{H}_c )$$

$$= \frac{1}{2} ( \tilde{\omega} \cdot \vec{H}_c + \tilde{H}_c \cdot \tilde{\omega} + \overleftrightarrow{I} \cdot \tilde{\omega}^t \cdot \vec{\omega} ) \tag{A. 41}$$

$$= \vec{0}$$

Note that the first term on the right of this equation is the negative of the second term; the third term is zero since $\tilde{\omega} \cdot \vec{\omega} = \vec{\omega} \times \vec{\omega} = \vec{0}$. Thus equation (A. 40a) is verified. Similarly

$$\overleftrightarrow{C}^t \cdot \vec{\omega} = \frac{1}{2} ( \tilde{\omega}^t \cdot \vec{\omega} + \overleftrightarrow{J} \cdot \tilde{H}_c^t \cdot \vec{\omega} + \overleftrightarrow{J} \cdot \tilde{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega} )$$

$$= \frac{1}{2} \overleftrightarrow{J} \cdot ( \tilde{H}_c^t \cdot \vec{\omega} + \tilde{\omega} \cdot \vec{H}_c ) \tag{A. 42}$$

$$= \overleftrightarrow{J} \cdot \tilde{H}_c^t \cdot \vec{\omega}$$

Thus equation (A. 40b) is also verified.

The relationship $\dot{\bar{\mu}} = \bar{\gamma} \cdot \bar{\mu} + \bar{\mu} \cdot \bar{\gamma}^t$ yields the requirement

$$\begin{aligned} \overleftrightarrow{I} &= \overleftrightarrow{C} \cdot \overleftrightarrow{I} + \overleftrightarrow{I} \cdot \overleftrightarrow{C}^t \\ &= \overleftrightarrow{D} + \overleftrightarrow{D}^t \end{aligned} \tag{A.43}$$

But from equation (A.37) it follows that

$$\overleftrightarrow{D} + \overleftrightarrow{D}^t = \tilde{\omega} \cdot \overleftrightarrow{I} + \overleftrightarrow{I} \cdot \tilde{\omega}^t \tag{A.44}$$

The right hand side of equation (A.44) is known to be equal to $\dot{\overleftrightarrow{I}}$. Hence equation (A.43) is also verified.

## B. INTERCONNECTING TWO RIGID BODIES

When a system of bodies is interconnected the primitive system equations are obtained by simply stacking all the individual bodies equations. Thus, for a two body system the equation $G = \mu \cdot g$ becomes

$$\begin{bmatrix} \vec{H}^1_{c_1} \\ \vec{H}^2_{c_2} \\ \vec{P}^1 \\ \vec{P}^2 \end{bmatrix} = \begin{bmatrix} \overleftrightarrow{I}^1 & & & \\ & \overleftrightarrow{I}^2 & & \\ & & \overleftrightarrow{M}^1 & \\ & & & \overleftrightarrow{M}^2 \end{bmatrix} \cdot \begin{bmatrix} \vec{\omega}^1 \\ \vec{\omega}^2 \\ \vec{v}_{c_1} \\ \vec{v}_{c_2} \end{bmatrix} \tag{B.1}$$

where the off-diagonal terms of $\mu$ are zero dyadics $\overleftrightarrow{O}$. Note that superscipts 1 and 2 are used to refer to bodies 1 and 2, and subscripts are used to refer to the center of mass points $c_1$ and $c_2$. The momentum formulation equation $\dot{G} - \gamma \cdot G = K$ is given by

$$\begin{bmatrix} \dot{\vec{H}}^1_{c_1} \\ \dot{\vec{H}}^2_{c_2} \\ \dot{\vec{P}}^1 \\ \dot{\vec{P}}^2 \end{bmatrix} = \begin{bmatrix} \vec{L}^1_{c_1} \\ \vec{L}^2_{c_2} \\ \vec{F}^1 \\ \vec{F}^2 \end{bmatrix} \tag{B.2}$$

and the velocity formulation equation $\mu \cdot \dot{g} + \gamma^t \cdot g = K$ is given by

$$\begin{bmatrix} \overleftrightarrow{I}^1 & & & \\ & \overleftrightarrow{I}^2 & & \\ & & \overleftrightarrow{M}^1 & \\ & & & \overleftrightarrow{M}^2 \end{bmatrix} \cdot \begin{bmatrix} \dot{\vec{\omega}}^1 \\ \dot{\vec{\omega}}^2 \\ \dot{\vec{v}}_{c_1} \\ \dot{\vec{v}}_{c_2} \end{bmatrix} + \begin{bmatrix} \overleftrightarrow{C}^{1^t} & & & \\ & \overleftrightarrow{C}^{2^t} & & \\ & & \overleftrightarrow{O} & \\ & & & \overleftrightarrow{O} \end{bmatrix} \cdot \begin{bmatrix} \vec{\omega}^1 \\ \vec{\omega}^2 \\ \vec{v}_{c_1} \\ \vec{v}_{c_2} \end{bmatrix} = \begin{bmatrix} \vec{L}^1_{c_1} \\ \vec{L}^2_{c_2} \\ \vec{F}^1 \\ \vec{F}^2 \end{bmatrix} \tag{B.3}$$

## B.1. Introduction of Relative Velocities

The above primitive system equations have 12 degrees of freedom. But in a system of two bodies it is usually convenient to model the second body with less than 6 degrees of freedom. In order to impose restrictions on the relative motion of the second body it is convenient to first transform the primitive variables and equations to new variables and equations so that the impositions of relative motion constraints becomes simpler.

Let $h_2$ be a point fixed in body 2 and let $b_2$ be a point fixed in body 1. The position vector to $h_2$ from $b_2$ is denoted by $\vec{R}_{h_2 b_2}$. Let $\overset{1}{\vec{R}}_{h_2 b_2}$ be the time derivative of $\vec{R}_{h_2 b_2}$ with respect to body 1. Then the inertial time derivative $\dot{\vec{R}}_{h_2 b_2}$ can be written as

$$\dot{\vec{R}}_{h_2b_2} = \overset{1}{\vec{R}}_{h_2b_2} + \vec{\omega}^1 \times \vec{R}_{h_2b_2} \tag{B.4a}$$

In order to put bodies 1 and 2 on a similar footing, let $h_1$ be a point fixed in body 1 and let $b_1$ be at the inertial reference frame origin. For convenience, this origin can be defined by an infinitesimal body 0 with an angular velocity $\vec{\omega}^0 = \vec{0}$. Then analogous to equation (B.4a) there is the result

$$\dot{\vec{R}}_{h_1b_1} = \overset{0}{\vec{R}}_{h_1b_1} + \vec{\omega}^0 \times \vec{R}_{h_1b_1} \tag{B.4b}$$

Now introduce the notation $\vec{V}^2$ and $\vec{V}^1$ as follows:

$$\vec{U}^2 = \overset{1}{\vec{R}}_{h_2b_2} \tag{B.5a}$$

$$\vec{U}^1 = \overset{0}{\vec{R}}_{h_1b_1} = \dot{\vec{R}}_{h_1b_1} = \dot{\vec{r}}_{h_1} = \vec{v}_{h_1} \tag{B.5b}$$

The additional equalities in equation (B.5b) follow from (B.4b) with $\vec{\omega}^0 = \vec{0}$ and from $\vec{R}_{h_1b_1} = \vec{r}_{h_1}$ since $b_1$ is at the inertial origin. The symbol $\vec{r}_{h_1}$, of course, denotes the position vector to the point $h_1$ from the inertial origin; the inertial time derivative of $\vec{r}_{h_1}$ is $\vec{v}_{h_1}$, the linear velocity of point $h_1$.

The relative angular velocities $\vec{\Omega}^2$ and $\vec{\Omega}^1$ are now introduced as follows:

$$\vec{\Omega}^2 = \vec{\omega}^2 - \vec{\omega}^1 \tag{B.6a}$$

$$\vec{\Omega}^1 = \vec{\omega}^1 - \vec{\omega}^0 = \vec{\omega}^1 \tag{B.6b}$$

The velocity transformation equation $g = A \cdot \bar{g}$ now takes the form

$$\begin{bmatrix} \vec{\omega}^1 \\ \vec{\omega}^2 \\ \vec{v}_{c_1} \\ \vec{v}_{c_2} \end{bmatrix} = \begin{bmatrix} \overleftrightarrow{E} & & & \\ \overleftrightarrow{E} & \overleftrightarrow{E} & & \\ \tilde{R}^t_{c_1h_1} & & \overleftrightarrow{E} & \\ \tilde{R}^t_{c_2h_1} & \tilde{R}^t_{c_2h_2} & \overleftrightarrow{E} & \overleftrightarrow{E} \end{bmatrix} \cdot \begin{bmatrix} \vec{\Omega}^1 \\ \vec{\Omega}^2 \\ \vec{U}^1 \\ \vec{U}^2 \end{bmatrix} \qquad \text{(B. 7)}$$

The inverse velocity transformation $\bar{g} = B \cdot g$ is given by

$$\begin{bmatrix} \vec{\Omega}^1 \\ \vec{\Omega}^2 \\ \vec{U}^1 \\ \vec{U}^2 \end{bmatrix} = \begin{bmatrix} \overleftrightarrow{E} & & & \\ -\overleftrightarrow{E} & \overleftrightarrow{E} & & \\ \tilde{R}_{c_1h_1} & & \overleftrightarrow{E} & \\ -\tilde{R}_{c_1h_2} & \tilde{R}_{c_2h_2} & -\overleftrightarrow{E} & \overleftrightarrow{E} \end{bmatrix} \cdot \begin{bmatrix} \vec{\omega}^1 \\ \vec{\omega}^2 \\ \vec{v}_{c_1} \\ \vec{v}_{c_2} \end{bmatrix} \qquad \text{(B. 8)}$$

The elements of A and B which are not shown are zero dyadics $\overleftrightarrow{O}$. The equation for $\vec{v}_{c_1}$ and $\vec{v}_{c_2}$ follow from taking the inertial time derivatives of the equations

$$\vec{r}_{c_1} = \vec{R}_{c_1h_1} + \vec{r}_{h_1} \tag{B.9a}$$

$$\vec{r}_{c_2} = \vec{R}_{c_2h_2} + \vec{R}_{h_2b_2} + \vec{R}_{b_2h_1} + \vec{r}_{h_1} \tag{B.9b}$$

and then making use of the facts that $\vec{R}_{c_1h_1}$ and $\vec{R}_{b_2h_1}$ are fixed in body 1 whereas $\vec{R}_{c_2h_2}$ is fixed in body 2.

## B.2. Transformed Variables and Equations

Now that the transformation operator A has been determined the new momentum and force variables can be determined via $A^t$. Thus, the equation $\overline{G} = A^t \cdot G$ yields

$$\overline{G} = A^t \cdot G = \begin{bmatrix} \overleftrightarrow{E} & \overleftrightarrow{E} & \tilde{R}_{c_1h_1} & \tilde{R}_{c_2h_1} \\ & \overleftrightarrow{E} & & \tilde{R}_{c_2h_2} \\ & & \overleftrightarrow{E} & \overleftrightarrow{E} \\ & & & \overleftrightarrow{E} \end{bmatrix} \cdot \begin{bmatrix} \vec{H}^1_{c_1} \\ \vec{H}^2_{c_2} \\ \vec{P}^1 \\ \vec{P}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{H}^1_{c_1} + \vec{H}^2_{c_2} + \tilde{R}_{c_1h_1} \cdot \vec{P}^1 + \tilde{R}_{c_2h_1} \cdot \vec{P}^2 \\ \vec{H}^2_{c_2} + \tilde{R}_{c_2h_2} \cdot \vec{P}^2 \\ \vec{P}^1 + \vec{P}^2 \\ \vec{P}^2 \end{bmatrix} \tag{B.10}$$

Now introduce the following notation

$$\vec{H}^{1}_{h_1} = \vec{H}^{1}_{c_1} + \tilde{R}_{c_1 h_1} \cdot \vec{P}^{1} \tag{B.11a}$$

$$\vec{H}^{2}_{h_1} = \vec{H}^{2}_{c_2} + \tilde{R}_{c_2 h_1} \cdot \vec{P}^{2} \tag{B.11b}$$

$$\vec{H}^{\bar{1}}_{h_1} = \vec{H}^{1}_{h_1} + \vec{H}^{2}_{h_1} \tag{B.11c}$$

$$\vec{H}^{2}_{h_2} = \vec{H}^{2}_{c_2} + \tilde{R}_{c_2 h_2} \cdot \vec{P}^{2} \tag{B.11d}$$

$$\vec{P}^{\bar{1}} = \vec{P}^{1} + \vec{P}^{2} \tag{B.11e}$$

For i and j equal to 1 and 2, $\vec{H}^{i}_{h_j}$ is the angular momentum of body i about the point $h_j$. $\vec{H}^{\bar{1}}_{h_1}$ is the angular momentum of the system of two bodies about the point $h_1$, and $\vec{P}^{\bar{1}}$ is the linear momentum of the system of two bodies. Combining equations (B.10) and (B.11) now yields

$$\bar{G} = \begin{bmatrix} \vec{H}^{\bar{1}}_{h_1} \\ \vec{H}^{2}_{h_2} \\ \vec{P}^{\bar{1}} \\ \vec{P}^{2} \end{bmatrix} \tag{B.12}$$

The equation $\overline{K} = A^t \cdot K$ yields similarly

$$\overline{K} = \begin{bmatrix} \vec{L}^{\overline{1}}_{h_1} \\ \vec{L}^{2}_{h_2} \\ \vec{F}^{\overline{1}} \\ \vec{F}^{2} \end{bmatrix} \tag{B.13}$$

where $\vec{L}^{\overline{1}}_{h_1}$ is the torque on the system of two bodies about $h_1$, $\vec{L}^{2}_{h_2}$ is the torque on body 2 about $h_2$, $\vec{F}^{\overline{1}}$ is the force on the system of two bodies, and $\vec{F}^{2}$ is the force on body 2.

Expanding the equation $\overline{\mu} = A^t \cdot \mu \cdot A$ yields

$$\overline{\mu} = \begin{bmatrix} \overleftrightarrow{I}^{\overline{1}}_{h_1h_1} & \overleftrightarrow{I}^{2}_{h_1h_2} & M^{\overline{1}}\tilde{R}_{c_{\overline{1}}h_1} & M^{2}\tilde{R}_{c_2h_1} \\ \overleftrightarrow{I}^{2}_{h_2h_1} & \overleftrightarrow{I}^{2}_{h_2h_2} & M^{2}\tilde{R}_{c_2h_2} & M^{2}\tilde{R}_{c_2h_2} \\ M^{\overline{1}}\tilde{R}_{c_{\overline{1}}h_1} & M^{2}\tilde{R}^{t}_{c_2h_2} & \overleftrightarrow{M}^{\overline{1}} & \overleftrightarrow{M}^{2} \\ M^{2}\tilde{R}^{t}_{c_2h_1} & M^{2}\tilde{R}^{t}_{c_2h_2} & \overleftrightarrow{M}^{2} & \overleftrightarrow{M}^{2} \end{bmatrix} \tag{B.14}$$

where for i, j, k equal to 1 and 2,

$$\overleftrightarrow{I}^{i}_{h_j h_k} = \overleftrightarrow{I}^{i} + M^{i}\,\tilde{R}_{c_i h_j} \cdot \tilde{R}^{t}_{c_i h_k} \tag{B.15a}$$

$$\overleftrightarrow{I}^{\bar{1}}_{h_1 h_1} = \overleftrightarrow{I}^{1}_{h_1 h_1} + \overleftrightarrow{I}^{2}_{h_1 h_1} \tag{B.15b}$$

$$\overleftrightarrow{M}^{\bar{1}} = \overleftrightarrow{M}^{1} + \overleftrightarrow{M}^{2} \tag{B.15c}$$

$$M^{\bar{1}}\,\tilde{R}_{c_{\bar{1}} h_1} = M^{1}\,\tilde{R}_{c_1 h_1} + M^{2}\,\tilde{R}_{c_2 h_1} \tag{B.15d}$$

$\overleftrightarrow{I}^{\bar{1}}_{h_1 h_1}$ is the inertia dyadic of the system of two bodies about the point $h_1$, and $\overleftrightarrow{M}^{\bar{1}}$ is the mass dyadic for the system of two bodies. The point $c_{\bar{1}}$ is the center of mass of the system of two bodies, and $\vec{R}_{c_{\bar{1}} h_1}$ is the position vector to $c_{\bar{1}}$ from $h_1$.

Expanding the equation $\bar{\nu} = B \cdot \nu \cdot B^{t}$ yields

$$\bar{\nu} = \begin{bmatrix} \overleftrightarrow{J}^{1} & -\overleftrightarrow{J}^{1} & \overleftrightarrow{Z}^{1}_{h_1} & -\overleftrightarrow{Z}^{1}_{h_2} \\ -\overleftrightarrow{J}^{1} & (\overleftrightarrow{J}^{2} + \overleftrightarrow{J}^{1}) & -\overleftrightarrow{Z}^{1}_{h_1} & (\overleftrightarrow{Z}^{2}_{h_2} + \overleftrightarrow{Z}^{1}_{h_2}) \\ \overleftrightarrow{Z}^{1\,t}_{h_1} & -\overleftrightarrow{Z}^{1\,t}_{h_1} & \overleftrightarrow{W}^{1}_{h_1 h_1} & -\overleftrightarrow{W}^{1}_{h_1 h_2} \\ -\overleftrightarrow{Z}^{1\,t}_{h_2} & (\overleftrightarrow{Z}^{2\,t}_{h_2} + \overleftrightarrow{Z}^{1\,t}_{h_2}) & -\overleftrightarrow{W}^{1}_{h_2 h_1} & (\overleftrightarrow{W}^{2}_{h_2 h_2} + \overleftrightarrow{W}^{1}_{h_2 h_2}) \end{bmatrix} \tag{B.16}$$

where for i, j, k equal to 1 and 2,

$$\overleftrightarrow{Z}^{i}_{h_j} = \overleftrightarrow{J}^{i} \cdot \tilde{R}^{t}_{c_i h_j} \tag{B.17a}$$

$$\overleftrightarrow{W}^{i}_{h_j h_k} = \overleftrightarrow{W}^{i} + \tilde{R}_{c_i h_j} \cdot \overleftrightarrow{J}^{i} \cdot \tilde{R}^{t}_{c_i h_k} \tag{B.17b}$$

There remains the evaluation of the term $\overline{X} = -\overline{\gamma} \cdot \overline{G}$ where $\overline{\gamma} = A^t \cdot \gamma \cdot B^t + A^t \cdot B^t$. When the reduced form of $\overline{\gamma}$ is used (i.e., when $\overline{X}$ is expressed as $\overline{X} = -\overline{\gamma}_{\overline{G}} \cdot \overline{G}$ ) the result is

$$\overline{X} = \begin{bmatrix} \tilde{v}_{h_1} \cdot \vec{P}^{1} \\ \tilde{v}_{h_2} \cdot \vec{P}^{2} \\ \vec{0} \\ \vec{0} \end{bmatrix} \tag{B.18}$$

Consequently, the momentum formulation equation of motion, $\dot{\overline{G}} + \overline{X} = \overline{K}$, takes the form

$$\begin{bmatrix} \dot{\vec{H}}^{\bar{1}}_{h_1} \\ \dot{\vec{H}}^{2}_{h_2} \\ \dot{\vec{P}}^{\bar{1}} \\ \dot{\vec{P}}^{2} \end{bmatrix} + \begin{bmatrix} \tilde{v}_{h_1} \cdot \vec{P}^{\bar{1}} \\ \tilde{v}_{h_2} \cdot \vec{P}^{2} \\ \vec{0} \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \vec{L}^{\bar{1}}_{h_1} \\ \vec{L}^{2}_{h_2} \\ \vec{F}^{\bar{2}} \\ \vec{F}^{2} \end{bmatrix} \tag{B.19}$$

This result is well known since it is an immediate consequence of the fundamental result

$$\dot{\vec{H}}^{A}_{a} + \vec{v}_a \times \vec{P}^{A} = \vec{L}^{A}_{a} \tag{B.20a}$$

$$\dot{\vec{P}}^{A} = \vec{F}^{A} \tag{B.20b}$$

where, given a system A and an arbitrary point a, $\vec{H}^{A}_{a}$ is the angular momentum of A about a, $\vec{L}^{A}_{a}$ is the torque on A about a, $\vec{P}^{A}$ is the linear momentum of A, and $\vec{F}^{A}$ is the force on A; $\vec{v}_a$ is the linear velocity of the point a. Equation (B.20) is used twice in equation (B.19): once with $A = \bar{1}$ and $a = h_1$, and once with $A = 2$ and $a = h_2$. In the first case system A is the system of two bodies; in the second case system A is body 2.

Even though the result (B.19) is well known, equations (B.14) for $\bar{\mu}$ and (B.16) for $\bar{\nu}$ are not well known. In fact, these equations have only been published in connection with the author's works on the transformation operator approach to multi-body dynamics [3], [4], [53].

## B. 3. Imposition of Constraints

Now suppose that body 2 has only a rotational degree of freedom relative to body 1. Then $\vec{U}^2$ is not a free variable; i.e., $\vec{U}^2$ is some prescribed function of time (perhaps $\vec{U}^2 = \vec{0}$ and the points $h_2$ and $b_2$ start out and remain coincident). Therefore $\vec{P}^2$ is not a free variable either, since $\vec{P}^2$ can be determined algebraically from the remaining free variables. Consequently the last equation of equations (B.19) can be removed from the set.

It is of interest to relate the free and constrained variables via the projectors $\pi^i = A_i \cdot B_i$ for $i = 1$ and 2. The free velocity is $\bar{g}_1$ and the constrained velocity is $\bar{g}_2 = \vec{U}^2$;

$$\bar{g}_1 = \begin{bmatrix} \vec{\Omega}^1 \\ \vec{\Omega}^2 \\ \vec{U}^1 \end{bmatrix} \tag{B.21}$$

$B_1$ is now determined from

$$\bar{g}_1 = \begin{bmatrix} \vec{\Omega}^1 \\ \vec{\Omega}^2 \\ \vec{V}^1 \end{bmatrix} = \begin{bmatrix} \overleftrightarrow{E} & & & \\ -\overleftrightarrow{E} & \overleftrightarrow{E} & & \\ \tilde{R}_{c_1 h_1} & & \overleftrightarrow{E} & \overleftrightarrow{O} \end{bmatrix} \cdot \begin{bmatrix} \vec{\omega}^1 \\ \vec{\omega}^2 \\ \vec{v}_{c_1} \\ \vec{v}_{c_2} \end{bmatrix} = B_1 \cdot g \tag{B.22a}$$

Similarly $B_2$ is determined from

$$\bar{g}_2 = \vec{U}^2 = [\, -\tilde{R}_{c_1h_2} \quad \tilde{R}_{c_2h_2} \quad -\overleftrightarrow{E} \quad \overleftrightarrow{E} \,] \cdot \begin{bmatrix} \vec{\omega}^1 \\ \vec{\omega}^2 \\ \vec{v}_{c_1} \\ \vec{v}_{c_2} \end{bmatrix} = B_2 \cdot g \tag{B.22b}$$

Now $g^1$ is the part of $g$ due to $\bar{g}_1$. Hence

$$g^1 = \begin{bmatrix} \overleftrightarrow{E} & & \\ \overleftrightarrow{E} & \overleftrightarrow{E} & \\ \tilde{R}^t_{c_1h_1} & & \overleftrightarrow{E} \\ \tilde{R}^t_{c_2h_1} & \tilde{R}^t_{c_2h_2} & \overleftrightarrow{E} \end{bmatrix} \cdot \begin{bmatrix} \vec{\Omega}^1 \\ \vec{\Omega}^2 \\ \vec{U}^1 \end{bmatrix} = A_1 \cdot \bar{g}_1 \tag{B.23a}$$

Similarly, $g^2$ is the part of $g$ due to $\bar{g}_2$ :

$$g^2 = \begin{bmatrix} \overleftrightarrow{O} \\ \overleftrightarrow{O} \\ \overleftrightarrow{O} \\ \overleftrightarrow{E} \end{bmatrix} \cdot \vec{U}^2 = A_2 \cdot \bar{g}_2 \tag{B.23b}$$

From equations (B.23) it follows that

$$g^1 = \begin{bmatrix} \vec{\omega}^1 \\ \vec{\omega}^2 \\ \vec{v}_{c_1} \\ \vec{v}'_{c_2} \end{bmatrix}, \quad \text{where } \vec{v}'_{c_2} = \vec{v}_{c_2} - \vec{U}^2 \tag{B.24a}$$

and

$$g^2 = \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vec{0} \\ \vec{U}^2 \end{bmatrix} \tag{B.24b}$$

Evidently, $g^2$ is restricted to a 3-dimensional subspace of the 12-dimensional space which contains $g$.

From the above expression for $A_i$ and $B_i$ there now follow the expression

$$\pi^1 = A_1 \cdot B_1 = \begin{bmatrix} \overleftrightarrow{E} & & & \\ & \overleftrightarrow{E} & & \\ & & \overleftrightarrow{E} & \\ \tilde{R}_{c_1 h_2} & -\tilde{R}_{c_2 h_2} & \overleftrightarrow{E} & \overleftrightarrow{O} \end{bmatrix} \tag{B.25a}$$

$$\pi^2 = A_2 \cdot B_2 = \begin{bmatrix} \overleftrightarrow{O} & & & \\ & \overleftrightarrow{O} & & \\ & & \overleftrightarrow{O} & \\ -\tilde{R}_{c_1 h_2} & \tilde{R}_{c_2 h_2} & -\overleftrightarrow{E} & \overleftrightarrow{E} \end{bmatrix} \tag{B.25b}$$

Using these expressions for the projectors, it is readily verified that

$$g^1 = \pi^1 \cdot g \tag{B.26a}$$

$$g^2 = \pi^2 \cdot g \tag{B.26b}$$

Note that the subspace to which $g^i$ is restricted is evident from the form of the projector $\pi^i$.

The free force is $\bar{K}_1$ and the constraint force is $\bar{K}_2$. For $i = 1$ and 2, $K^i$ is the part of K due to $\bar{K}_i$. $K^1$ and $K^2$ are given as follows

$$K^1 = \pi^{1^t} \cdot K = \begin{bmatrix} \vec{L}^1_{c_1} + \tilde{R}^t_{c_1 h_2} \cdot \vec{F}^2 \\ \vec{L}^2_{c_2} - \tilde{R}^t_{c_2 h_2} \cdot \vec{F}^2 \\ \vec{F}^1 + \vec{F}^2 \\ \vec{0} \end{bmatrix} \tag{B.27a}$$

$$K^2 = \pi^{2^t} \cdot K = \begin{bmatrix} -\tilde{R}^t_{c_1 h_2} \cdot \vec{F}^2 \\ \tilde{R}^t_{c_2 h_2} \cdot \vec{F}^2 \\ -\vec{F}^2 \\ \vec{F}^2 \end{bmatrix} \qquad \text{(B.28b)}$$

From these expression for $K^1$ and $K^2$ it is evident that $K^1$ is restricted to a 9-dimensional subspace whereas $K^2$ is in a 3-dimensional subspace.

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